Vol. 30, No. 4, March-April, 1957

MATHEMATICS magazine

MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders

Address editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to a member of the executive committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are 1 yr. $3.00; 2 yrs. $5.75; 3 yrs. $8.50; 4 yrs. $11.00; 5 yrs. $13.00. Sponsoring subscriptions are $10.00; single copies 65¢, Reprints, bound 1¢ per page plus 10¢ each, provided your order is placed before your article goes to press.

Subscriptions and other business correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, California.

Entered as second-class matter March 23, 1948 at the Post Office Pacoima, California, under act of Congress of March 3, 1879.

MATHEMATICS MAGAZINE

Vol. 30, No. 4, March-April 1957

CONTENTS

OUR CONTRIBUTORS

AUTHORS FROM PRECEEDING ISSUES :

A. RAJAGOPAL is a student at Lingaraj College, Belgaum, India. He passed the Secondary School Certificate Examination, University of Mysore, 1955, I class. His interests lie in the fields of physics and mathematics.

HIROYOSHI SASAYAMA, Assistant Professor in the Faculty of Science and Engineering, Meijo University, was born on February 2, 1929 in Tokyo, and received his formal education in Sendai (B.S. in Mathematics, Tohoku University '51). Before coming to Meijo University, Professor Sasayama held positions at the Faculty of Science, Tohoku University. His non-mathematical interests include the music of Mozart and the collection of rare books.

EDITORIAL NOTES:

Authors, please keep duplicates of your papers. From now on none will be returned unless we desire changes or postage is enclosed. (Our annual postage exceeds the entire cost of printing one issue.)

Libraries, please note that this magazine does not appear in July-Aug., i.e. there is no "Number 6", and that each volume index appears in the first issue of the next volume (loose in the future).

ON EXTENSORS IN THE CALCULUS OF VARIATIONS

Homer V. Craig

The purpose of this note is to point out that a merely superficial examination of some of the simpler problems of the calculus of variations will suffice: (1) to reveal that the primary extensors associated with the integrand functions are involved, and (2) to indicate an examination of the geometric entities (parameterized arcs, surfaces, etc.) determined by equating or linearly relating these extensors. This done it then becomes obvious in the case of the simpler problems that these entities are the "extremals" or "stationary" arcs and surfaces. We do not, however, mean to imply that the classical procedure based on integration by parts and the fundamental lemma should be discarded, instead our objective is to indicate that the components of the primary extensors appear as elements or building blocks. The role of the tensors of course becomes apparent since they are carried along with the extensors.

A knowledge of advanced calculus (including in particular the chain rule of differentiation, the concepts single and multiple integrals, and the rule for differentiating integrals with fixed limits) is necessary for reading the paper without preliminary study. Included in the expository material are the *extended coordinate transformation*, the *tensor* and *extensor transformation equations*, the notions *variation function* and *stationary arc*, certain of the simpler problems of the calculus of variations, and the celebrated *Euler equations*.

1. Notational preliminaries.

We shall formulate the problems to be considered in terms of N variables $x^1, x^2, \ldots, x^N$ with the superscripts used as marks of distinction rather than as indicators of powers so that x^1 (read "x upper one") and x^2 may be regarded as essentially two different letters - as a matter of fact, for $N = 2$, x^1 and x^2 might be identified with the familiar variables x and y of a rectangular Cartesian coordinate system. In general, x^a will represent variable number a of the set of variables of a certain coordinate system (x). Similarly, $\bar{x}^r$ will denote variable number r of a second coordinate system $(\bar{x})$. Usually, we shall distinguish symbols bearing indices (subscripts and superscripts) which belong to a coordinate system (x) from their mates which belong to a second system $(\bar{x})$ by reserving indices at the first of the alphabet (a, b, c, d) for system (x) and those near

the last $(r, s, t, u,)$ for $(\bar{x})$. The symbol $x^a(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ will be used to denote the value of x^a in terms of the coordinate variables of the system $(\bar{x})$, and thus connects the two systems and does not belong exclusively to one or the other. This symbol is of course a function symbol and may be read "the x-upper-a function of the $\bar{x}$'s." We shall subsequently abridge it to $x^a(\bar{x})$ with the unadorned $\bar{x}$ denoting the entire set of variables $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N$.

Coordinate transformations and extended coordinate transformations. In terms of the notation just introduced, the coordinate transformations which we shall have occasion to use may be written in the form
(1.1)
$$x^a = x^a(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N) = x^a(\bar{x}); \quad \bar{x}^r = \bar{x}^r(x^1, x^2, \ldots, x^N) = \bar{x}^r(x).$$

These equations (under restrictions if necessary as to the size of the variables) are to have the following properties: (1) the functions involved are not only single-valued and continuous but have continuous derivatives of all orders up to and including the Mth, i.e., they are of class C^M; and (2) the substitution from either set into any member of the other set yields an identity, in other words, either set of equations is a solution of the other set.

If we now make each of the N variables of say system (x) a class C^M function $(M \geq 1)$ of an auxilliary variable t, $[x^a = x^a(t)]$ then each of the $\bar{x}$'s becomes a class C^M function of t. We may then differentiate (1.1) M times with respect to t. The resulting equations together with the original set (1.1) make up what is called the *extended coordinate transformation*. To simplify the writing, we denote derivatives with respect to the parameter t by means of primes and enclosed Greek indices, thus

$$x^{a\prime} = x'^a = dx^a/dt, \quad x^{a\prime\prime} = x''^a = d^2x^a/dt^2, \quad x^{a(\alpha)} = d^\alpha x^a/dt^\alpha.$$

In addition, we adopt the notation

$$X^{\alpha a}_{\rho r} = \partial x^{(\alpha)a}/\partial \bar{x}^{(\rho)r}, \quad X^{\rho r}_{\alpha a} = \partial \bar{x}^{(\rho)r}/\partial x^{(\alpha)a},$$

$$X^{0a}_{0r} = X^a_r = \partial x^a/\partial \bar{x}^r, \quad X^a_{rs} = \partial^2 x^a/\partial \bar{x}^r \partial \bar{x}^s$$

and agree that repeated lower case Latin indices are to be assigned the values 1, 2, ..., N and the resulting terms added. Similarly, repeated lower case Greek indices generate sums 0 to M. In this symbolism the extended coordinate transformation for $M = 2$ is as follows:

$$x^a = x^a(\bar{x}), \qquad \bar{x}^r = \bar{x}^r(x),$$

(1.2)

$$x'^a = X^a_r \, \bar{x}'^r, \qquad \bar{x}'^r = X^r_a \, x'^a,$$

$$x''^{a} = X^{a}_{r}\,\bar{x}''^{r} + X^{a}_{rs}\,\bar{x}'^{r}\bar{x}'^{s}, \qquad \bar{x}''^{r} = X^{r}_{a}\,x''^{a} + X^{r}_{ab}\,x'^{a}x'^{b}.$$

Here we may note in passing that

$$(1.3)\qquad X^{1a}_{1r} = X^{a}_{r}, \quad X^{1r}_{1a} = X^{r}_{a}, \quad X^{\alpha a}_{\rho r} = 0 \qquad \text{if } \rho > \alpha\,.$$

The infinite collection of extended coordinate transformations of the type (1.2) for $M = 1$ is the fundamental set for the extensor theory used in this paper.

2. The extensor transformation equations.

The quantities which we call extensor components of the first order (for general, unspecified M) are characterized by the following qualifications: (1) there is *one set* of these quantities at a given point on a parameterized arc for *each coordinate system* of our fundamental collection; (2) there are $(M + 1)N$ of these quantities (the components) in each set and so they may be labelled $E^{\alpha a}$ or $E_{\alpha a}$ with α limited to the values $0, 1, \ldots, M$ and a to the values $1, 2, \ldots, N$; (3) they satisfy one or the other of the two different transformation laws:

$$(2.1)\qquad E^{\alpha a} = \bar{E}^{\rho r}\, X^{\alpha a}_{\rho r}, \qquad \bar{E}^{\rho r} = E^{\alpha a}\, X^{\rho r}_{\alpha a};$$

$$(2.2)\qquad E_{\alpha a} = \bar{E}_{\rho r}\, X^{\rho r}_{\alpha a}, \qquad \bar{E}_{\rho r} = E_{\alpha a}\, X^{\alpha a}_{\rho r}.$$

Equation (2.1) is described as *excontravariant* while (2.2) is said to be *excovariant*.

The most characteristic feature of the equations from the standpoint of their purely formal structure is that they may be generated from the skeleton equation $E = EX$ by the following procedure: (1) attach a doublet index (Greek-Latin pair) to the E on the left, (this pair will not only serve to number or identify the individual component but will constitute a coordinate marker, αa and ρr for example indicating coordinate systems (x) and $(\bar{x})$, respectively); (2) displace a duplicate doublet horizontally to the X (thus if superscript αa is added to the E on the left then αa is placed on the X in the superscript position); (3) fill in the blank position on the X with a doublet from the other half of the alphabet and repeat in the opposite position on E (thus if αa was used in steps (1) and (2) then any one of ρr, σs, τt for example may be used in step (3). The bars on the E's in (2.1) and (2.2) are superfluous but were included for the sake of emphasis.

With regard to an individual E such as E^{02} or E_{M2} the symbol as a whole represents, in the instances which we shall encounter, a *real number* called the value of the component. The doublet indices such as

$0\,2$ and $M\,2$ are identifying number pairs the first of which is called the rank. Thus E^{02} is the second component of rank zero, E^{12} is the second component of rank one, E_{M2} is the second component of rank M. The ranks associated with the minimum superscript value *zero* and the maximum subscript value M are called *tensor* ranks. The right members of the extensor transformation equations (2.1) may be partly expanded by assigning the repeated Greek index values from the set 0, 1, ..., M with the repeated Latin index left in literal form and calling for summation from *one* to N, or they may be expanded completely by assigning the repeated doublets the number pairs 0 1, 02, ..., $0N$; 11, 12, ..., $1N$; ..., $M1$, $M2$, ..., MN. The first method is quite useful at times and may be called summing by ranks. To illustrate this procedure, we write out (2.2) for $M=1$ taking into account equations (1.3), thus

$$E_{1a} = \overline{E}_{\rho r}\, X^{\rho r}_{1a} = \overline{E}_{0r}\, X^{0r}_{1a} + E_{1r}\, X^{1r}_{1a} = \overline{E}_{1r}\, X^{r}_{a}$$

$$E_{0a} = \overline{E}_{\rho r}\, X^{\rho r}_{0a} = \overline{E}_{0r}\, X^{r}_{a} + \overline{E}_{1r}\, X^{1r}_{0a}$$

The first of these equations is an instance of the *tensor* transformation law. Here we note that if the tensor components in any system $(\overline{x})$ all vanish, i.e., if $\overline{E}_{1r}=0$, then all of the tensor components in any other system (x) likewise vanish, i.e., $E_{1a}=0$. In this case the components of next rank (rank 0) become tensors. Symbolically, if $\overline{E}_{1r}=0$ and $M=1$, then $E_{0a}=\overline{E}_{0r}\,X^{r}_{a}$.

An additional property which is of fundamental importance is the invariance of extensor equations. For example, if E and F are two first order excovariant extensors and A and B are any two multipliers (constants or functions) then the quantities $AE_{\alpha a}+BF_{\alpha a}$ are extensor components and if $AE_{\alpha a}+BF_{\alpha a}=0$ at some point P of a certain arc C, for all admissible choices of α and a, then for $(\overline{x})$ any other coordinate system of the fundamental collection, $A\overline{E}_{\rho r}+B\overline{F}_{\rho r}$ likewise vanishes for P, C. Similarly, if the equations $AE_{\alpha a}+BF_{\alpha a}=0$ are likewise satisfied all along an arc C, then the mate equations in $(\overline{x})$, namely, $AE_{\rho r}+BF_{\rho r}=0$ are likewise satisfied along C. Thus if a set of curves is determined by setting $AE_{\alpha a}+BF_{\alpha a}$ equal to zero, then this set of curves is independent of the coordinate system. To prove these assertions it will suffice to multiply the equations

$$E_{\rho r} = E_{\alpha a}\, X^{\alpha a}_{\rho r}, \qquad F_{\rho r} = F_{\alpha a}\, X^{\alpha a}_{\rho r}$$

by A and B respectively and add, thereby obtaining the relationship

$$AE_{\rho r} + BF_{\rho r} = (AE_{\alpha a} + BF_{\alpha a})\, X^{\alpha a}_{\rho r}.$$

It is now apparent that if $AE_{\alpha a} + BF_{\alpha a} = 0$ for all pairs αa, then for any admissible values for ρ and r the left member is likewise zero.

3. Some examples of extensors.

Let us consider a function $F(x, x')$ of the coordinate variables and their derivatives with respect to a curve parameter t. Now if C is an arc, defined by $x^a = x^a(t)$, and of class C' for t in the interval (t_1, t_2), then substitution of $x^a(t)$ and $x'^a(t)$ for each x^a and x'^a in F will yield a function of t - "the value of F along C." If this is an integrable function then there is determined of course a definite numerical value for $\int_{t_1}^{t_2} F(x,x')dt$, and we may say that the integral is a function from a class C' parameterized *arc* to a *real number*. We next note that if we define a mate function $F(\bar{x},\bar{x}')$ for coordinate system $(\bar{x})$ by the simple and natural procedure of replacing the x's and x''s in $F(x,x')$ by their values in terms of the $\bar{x}$'s and $\bar{x}'$'s given by (1.2), or in symbols, if we define $\bar{F}$ by

$$\bar{F}(\bar{x},\bar{x}') \equiv F[x(\bar{x}),\ x'(\bar{x},\bar{x}')] \tag{3.2}$$

then along any arc C of class C' we have $\bar{F} \underset{t}{\equiv} F$. Now since the functions are identically equal in t along C their integrals along C are equal, i.e., $\int_{t_1}^{t_2} F\, dt = \int_{t_1}^{t_2} \bar{F}\, dt$.

If instead of evaluating the variables $\bar{x},\bar{x}'$ for a specific curve C, we leave them unspecified, then (3.2) remains an identity in the $\bar{x}$'s and $\bar{x}'$'s, or if we replace the variables $\bar{x},\bar{x}'$ by their values in terms of the x's and x''s through (1.2), then of course we have an identity in the latter variables. If we differentiate (3.2) with respect to $\bar{x}^{(\rho)r}$ (using the chain rule for the right member and employing the notation $\partial\bar{F}/\partial\bar{x}^{(\rho)r} = \bar{F}_{;\rho r}$, $F_{;\alpha a} = \partial F/\partial x^{(\alpha)a}$ there results the relationship

$$F_{;\rho r} = F_{;\alpha a} X^{\alpha a}_{\rho r}, \quad M = 1. \tag{3.3}$$

This equation asserts that the quantities $F_{;\rho r}$, $F_{;\alpha a}$ are extensor components.

The tensor member of this extensor is obtained, as we have noted before, by asigning the Greek subscript the maximum value M which is unity in this case. Thus we conclude that the quantities $\bar{F}_{;1r}$, $F_{;1a}$ are the components of a covariant tensor. But $F_{;\alpha a}$ is not the only extensor derivable from F by differentiation for there is a theorem (see [11] p. 22) which asserts that if the quantities T_a are the components of a covariant tensor and are of class C^M, then the quantities $E_{\alpha a}$ defined by

$$E_{\alpha a} = \binom{M}{A} T_{a}^{\cdot \cdot \cdot (M-A)}, \quad A = \alpha \tag{3.4}$$

(with similar definitions in the other coordinate systems) are the components of an extensor. In the right member of (3.4) we have substituted capital alpha A to suspend the summation convention. With $M = 1$, the two binomial coefficients $\binom{1}{0}$ and $\binom{1}{1}$ are each equal to one and may be omitted. Hence we may conclude that if F is of class $C^{M''}$ and if $F_{\alpha a}$ is defined by

$$F_{\alpha a} = F_{;1a}^{(1-\alpha)} \tag{3.5}$$

then the quantities $F_{\alpha a}$ are the components of an extensor. We refer to the two extensors represented by $F_{;\alpha a}$ and $F_{\alpha a}$ as the primary extensors associated with F because their components are pure rates of change. Perhaps, we should explain at this point that we speak of $F_{;\alpha a}$ for example as an extensor rather than as the components of an extensor in system (x) for economy of expression.

Concerning the two extensors $F_{\alpha a}, F_{;\alpha a}$ we note that they coincide in their tensor members for $F_{1a} = F^{(0)}_{;1a} = F_{;1a}$. Furthermore, since F_{0a} is $F'_{;1a}$, it is evident that excepting the very special case $F_{;1a;1b} \equiv 0$, F_{0a} involves the second derivatives x''. Consequently, we now raise our requirements by demanding that only functions F and arcs C which are of class C'' are to be regarded as admissible. However, it is urgent for the calculus of variations to put another requirement on F, namely, that it be such that $I(C)$,

$$I(C) = \int_{t_1}^{t_2} F(x,x')\,dt,$$

is independent of the choice of the parameter t. Specifically, this latter assumption means that the parameter transformation: $t = t(T)$, $T = T(t)$ with the functions of class C' and $dT/dt > 0$ in (t_1, t_2), when applied to the integral, leads to the relationship

$$\int_{t_1}^{t_2} F(x,\ dx/dt)\,dt = \int_{T(t_1)}^{T(t_2)} F(x,\ dx/dT)\,dT \tag{3.6}$$

It may be shown {see [1], pp.193-196} that this requirement implies that

$$x'^{a} F_{;1a} \equiv F. \tag{3.7}$$

If we differentiate (3.7) with respect to t, we learn that

$$x'^{a}(F_{;1a})' + x''^{a} F_{;1a} = x'^{a} F_{;0a} + x''^{a} F_{;1a}$$

or

$$x'^{a(\alpha)} F_{\alpha a} = x'^{a(\alpha)} F_{;\alpha a} \tag{3.8}$$

Now a theorem {see [2], p.766, or [3], p.218 or [11], p.22} concerning the manufacture of extensors from contravariant tensors V^a, asserts that if V^a is contravariant and of class C^M, then the quantity $V^{a(\alpha)}$ are the components of an extensor of the type $E^{\alpha a}$. Furthermore, the

equations (2.1) show that the x'^a are themselves contravariant and therefore the quantities $x'^{a(\alpha)}$ or x'^a, x''^a are extensor components. Sums such as are involved in the left and right members of (3.8) are called extensor contractions and are *invariants*. The equations (3.8) may be regarded as asserting that F must be such that its two associated extensors $F_{\alpha a}$ and $F_{;\alpha a}$ (which already coincide in their tensor members) must have equal contractions with the extensor $x'^{a(\alpha)}$.

The variation functions. So far we have had occasion to consider the function F either with its arguments unspecified or else with its arguments evaluated along a certain specific curve. A third alternative, which we shall need to consider in our calculus of variations problem, is to introduce a family of curves defined by a base curve C_0, a set of "*variation functions*" $V^a(t)$ and a "*variation parameter*" v.

Specifically, the assumptions to be made are as follows: First, we require that the function $F(x,x')$ be of class C'' for the x's in a region R and for all values of the x''s excepting $x'^1 = x'^2 = \ldots x'^N = 0$ which are excluded, and second that it satisfy (3.6). Next we assume that we have given a pair of points P_1 and P_2 in R and an arc C_0 of class C'' (given by $x^a = x^a(t)$ which lies in R and passes through P_1 and P_2 for $t = t_1$ and $t = t_2$, respectively, with $t_1 < t_2$. Finally, the functions $V^a(t)$ are to have the property that there is a number interval $i(v)$ containing zero such that for each fixed value of the real variable v in $i(v)$ the functions $x^a(t) + vV^a(t)$ meet the requirements imposed on $x^a(t)$. Thus these variation functions will be assumed to be of class C'' and they must vanish at t_1 and t_2. To shorten the writing, we shall use the word *admissible* to indicate that these conditions are in force, and shall denote the expression $x^a(t) + vV^a(t)$ by the symbol $x^a(t,v)$. Perhaps it should be mentioned at this point, that a more exhaustive treatment would require modification of these conditions.

Comparison curves. Obviously, the equations

$$x^a = x^a(t,v) = x^a(t) + vV^a(t) \tag{3.9}$$

define an admissiable arc $C(v)$ for each fixed value of v in $i(v)$. These arcs are called "*variation curves*" or "*comparison curves*." In particular the curve $C(0)$ defined by assigning v the value zero in (3.9) is the base curve, i.e., $C(0)$ is C_0. Evidently, if we replace each x in $F(x\ x')$ by $x^a(t,v)$ (noting of course that x'^a now becomes $\partial x^a(t,v)/\partial t = x'^a(t) + vV'^a$) then F by virtue of this substitution becomes a function of t and v which is of class c'' in v. The integral $\int_{t_1}^{t_2} F\, dt$ is now a class C'' function of v alone, which we denote by

$I(v) = \int_{t_1}^{t_2} F\,dt$, taken along $C(v)$.

To illustrate, let $N = 2$ and let F be the integrand for arc length in retangular Cartesian coordinates, $[(x')^2 + (y')^2]^{\frac{1}{2}}$ or in index notation $[(x'^1)^2 + (x'^2)^2]^{\frac{1}{2}}$. We note here that partial derivatives of F with respect to x' and y' are nonexistent for $x' = 0$, $y' = 0$ and we have excluded this set of values for the primes so that this particular and important integrand function would be admissible. The function $I(v)$ is now the arc length from P_1 to P_2 along the varied curve $C(v)$. We now select for the base curve C_0 the straight line joining P_1 and P_2 and accordingly fix $I(0)$ as the straight line distance from P_1 to P_2. Now if we assume for the moment that we already know that the straight line furnishes the shortest distance, then we may conclude that for any given set of admissible variation functions and each v in the associated $i(v)$, we have $I(v) > I(0)$. Here we note that it may be necessary to restrict the size of v to exclude the set of vanishing primes. Furthermore, since $I(v)$ is of class C'', the derivative dI/dv vanishes at $v = 0$ for any set of admissible variation functions $V^a(t)$. In other words the graph of $I(v)$ against v has a horizontal tangent at $v = 0$ and is concave upward at this point regardless of the choice of the admissible variation functions excluding the set $V^1(t) = 0$, $V^2(t) = 0$, of course.

With regard to the computation of dI/dv at $v = 0$, we note: first, that because the limits t_1 and t_2 on the integral are independent of v, we may proceed by passing over the integral sign and differentiating F; and second, that a derivative depends on the functional relationship or structure involved and the prescribed value of the independent or "differentiation" variable but not on the choice of symbol used for that variable, for example, $d(y^2)/dy$ at $y = a$ is $2a$ and $d(x^2)/dx$ at $x = a$ is $2a$. Thus if we replace the x's in $F(x,x')$ by the functions $x^a(t,v)$; select a value for t in (t_1, t_2), call it k, and assign v the value zero; then the partial derivative of F with respect to the symbol $x^a(t,v)$ evaluated at $x^a(k,0)$ is the same as the partial derivative of F (in its original form) with respect to x^a at $x^a = x^a(k)$ since according to (2.9) $x^a(k,0) = x^a(k)$. Here of course the arguments x' are to be the same in the two cases. Hence to evaluate dI/dv at $v = 0$, we pass over the integral sign and: (1) compute $\partial F(x,x')/\partial x^a$, (2) multiply the result by $\partial x^a(t,v)/\partial v$ or $V^a(t)$, (3) compute $\partial F(x,x')/\partial x'^a$, (4) multiply this last quantity by $V^{a\prime}$ since $\partial x'^a(t,v)/\partial v = V^{a\prime}$, and (5) add the products obtained in (2) and (4). In this way be obtain

$$(3.10)\quad \left.\frac{dI}{dv}\right|_{v=0} = \int_{t_1}^{t_2} (F_{;0a}V^a + F_{;1a}V^{a\prime})\,dt = \int_{t_1}^{t_2} F_{;\alpha a}V^{a(\alpha)}\,dt.$$

In order to express dI/dv at $v=0$ in terms of a second coordinate system $(\bar{x})$, we first substitute the right members of the equations (3.9) of the comparison curves, namely, $x^a(t) + vV^a(t)$ for the x's in the equations $\bar{x}^r = \bar{x}^r(x)$ of the set (1.2). This gives us the $\bar{x}$ equations of these curves, $\bar{x}^r = \bar{x}^r[x(t) + vV(t)]$ or briefly $\bar{x}^r = \bar{x}^r[x(t,v)]$. If we differentiate this last equation with respect to v, at $v=0$, note that $\partial x^a/\partial v = V^a$, and let $\bar{V}^r$ represent $\partial\bar{x}^r/\partial v$, at $v=0$, then we obtain the relationship

$$(3.11) \qquad \bar{V}^r = X^r_a V^a$$

Thus the quantities V^a, $\bar{V}^r$ are the components of a contravariant tensor and therefore $V^{a(\alpha)}$, $V^{r(\rho)}$ are extensor components. Thus examining

$$(3.12) \qquad \left.\frac{dI}{dv}\right|_{v=0} = \int_{t_1}^{t_2} F_{;\alpha a} V^{a(\alpha)} dt = \int_{t_1}^{t_2} \bar{F}_{;\rho r} \bar{V}^{r(\rho)} dt$$

we see that the integrand of dI/dv (at $v=0$) is an invariant given by the contraction of extensors. We observe in passing that the conditions $V^a(t_1) = V^a(t_2) = 0$ and (3.11) imply that the $\bar{V}^r(t)$ likewise vanish at t_1 and t_2. The fact that $\bar{x}^r(t,v)$ does not necessarily involve v linearly is of no importance to us.

4. Some simple calculus of variations problems.

It is obvious that if an arc C_0 furnishes an integral, of the type which we hade been considering, an extreme value (maximum or minimum) relative to the set of all admissible comparison curves, then dI/dv (at $v=0$) must vanish when C_0 is the base arc. Accordingly, the problem of determining the differential equations for these arcs is a fundamental one. These differential equations are called variously: the *Euler equations*, the *Euler-Lagrange equations*, and in mechanics the *Lagrange equations*. The associated integral curves are called *stationary* arcs or *extremals*. We now turn to the problem of finding the Euler eauations for one or two comparatively simple problems. Since we have already indicated in some detail the conditions to be imposed, we shall formulate these problems in an abridged form.

The Euler equations for $\int F(x,x')dt$. For an initial problem, we consider the matter of determining the Euler equations associated with the integral, I, studied in article 3. The usual method of attack {see [1] pages 1-27, 189-202} is to integrate one of the terms in equation (3.10) by parts, then prove a theorem (called the fundamental lemma) which enables one to discard the variation functions V^a and the integral sign and thereby obtain a set of differential equations (the Euler equations) expressed in terms of derivatives of F. Instead of following this classical procedure, which is presented in detail in

various works, we shall for the sake of variety, proceed heuristically, noting that the facts presented in article 3 almost force us to examine the equations

$$(4.1) \qquad F_{;\alpha a} = F_{\alpha a}, \qquad M = 1,$$

obtained by equating the primary extensors associated with F. Once this is done a very brief and easy argument will show that equations (4.1) do furnish solutions. But first let us review the facts concerning the primary extensors and the calculus of variations problem at hand. They are as follows:

(1) Since the differential equations sought must have a solution passing through two prescribed points which may be selected in infinitely many ways, they must be at least of second order and preferably no higher. The equations (4.1) are of second order except in the case $F_{;1a;1b} \equiv 0$.

(2) The extensors $F_{;\alpha a}$, $F_{\alpha a}$ already agree in their tensor rank ($\alpha = 1$) and because of the imposed invariance of $\int F dt$ under a parameter transformation, must be sufficiently alike to have equal contractions with the extensor $x'^{a(\alpha)}$.

(3) The value of dI/dv (at $v = 0$) for a given base curve C_0 is equal to the integral of the contraction of the extensors $F_{;\alpha a}$, $V^{a(\alpha)}$ and is therefore invariant under a coordinate transformation. Consequently, the curves C_0 determined by dI/dv (at $v = 0$) $= 0$ are independent of the coordinate system. On the other hand, if (4.1) in coordinate system (x) is satisfied by a certain curve C_0, then (4.1) in any other coordinate system $(\bar{x})$ of our collection, specifically (4.1) $\overline{F}_{;\rho r} = \overline{F}_{\rho r}$, is likewise satisfied by C_0.

Now, since $F_{;\alpha a}$ and $F_{\alpha a}$ are involved in the problem through their contractions with the extensor $x^{a(\alpha)}$ (see item (2) of the preceeding list); and since $F_{;\alpha a} V^{a(\alpha)}$ furnishes through its integral the value of dI/dv (at $v = 0$); it is natural to inquire about the contraction of the other extensor $F_{\alpha a}$ with $V^{a(\alpha)}$ and its integral. Examination shows at once that the invariant $F_{\alpha a} V^{a(\alpha)}$ is the derivative with respect to t of the invariant $F_{;\alpha a}V^{b}$ (a contraction of tensors). Thus if we recall that $V(t_1) = V(t_2) = 0$, then we have

$$(4.2) \qquad \begin{aligned} F_{\alpha a} V^{a(\alpha)} &= (F_{;1a} V^{a})'; \\ \int_{t_1}^{t_2} F_{\alpha a} V^{a(\alpha)} dt &= (F_{;1a} V^{a}) \Big|_{t_1}^{t_2} = 0. \end{aligned}$$

We now turn to the question of whether the curves which satisfy (4.1) also make dI/dv (at $v = 0$) vanish. A glance at (3.10), (4.1), and (4.2) will show that this is indeed the case. As a matter of fact

the equations (4.1) are essentially the Euler equations $F_{;0a} = (F_{;1a})'$ and so we see that these fundamental equations express the equality of the two extensors $F_{;aa}$ and F_{aa}. If we rewrite (4.1) in the pattern $F_{;aa} - F_{aa} = 0$, then we see that the left member is an extensor whose tensor rank vanishes identically and consequently the next rank (furnished by $\alpha = 0$) namely $F_{;0a} - (F_{;1a})'$ is a tensor which in turn vanishes along an extremal. The individual pieces of this expression, $F_{;0a}$ and $(F_{;1a})'$ are not themselves tensors and they do not have desirable transformation equations when isolated from the extensors to which they belong.

The foregoing approach to the Euler equations does not however show that the solutions of (4.1) are the only stationary curves. Apparently, the fundamental lemma is needed, but not integration by parts. Since the purpose of this paper is to examine the role of the primary extensors, we shall merely outline one method of attack.

Along any parameterized arc of class C'' the quantities $F_{;0a}$, $(F_{;1a})'$ (and therefore their differences) are functions of the parameter t. Hence if C_0 is a class C'' stationary curve* which perhaps does not satisfy the Euler equations, then along C_0, $F_{;0a} = (F_{;1a})' + M_a(t)$ with the functions M_a continuous. Substitution into (3.10) will show that dI/dv at $v = 0$) is now equal to $\int_{t_1}^{t_2} M_a V^a dt$. The fundamental lemma would now allow us to conclude that the M_a are identically zero.

Before turning to our next problem, we note that a more general procedure than equating extensors is to assume a linear relationship among them such as $AF_{;aa} + F_{aa} = 0$. Excluding the case $F_{;1a} \equiv 0$, we must have $A = -1$ for the tensor members to be satisfied.

The isoperimetric problem. This problem is similar to the preceding except that the base arcs $x^a = x^a(t)$ and the comparison curves $x^a = x^a(t,v)$ in addition to passing through two prescribed points as before are now required to give a second integral $\int_{t_1}^{t_2} G(x,x')dt$ a constant value K. Thus the question under consideration is to find the differential equations for base arcs which give $\int_{t_1}^{t_2} F(x,x')dt$ a stationary value relative to the set of comparison arcs which satisfy $\int G\, dt = K$. Thus we shall admit only such functions G and constants K as allow at least one family of one parameter comparison curves.

Apparently, as a result of the added condition, $\int G\, dt = K$, we should have three constants at our disposal where two sufficed in the previous problem.

The conditions to be imposed on the various functions are similar to those of the preceding problem. In particular, we require that

* Relative to the set of all admissible variation functions.

$$x^a(t,0) = x^a(t), \quad x^a(t_1,v) = x^a(t_1), \quad x^a(t_2,v) = x^a(t_2)$$

and that $\int F\,dt$ and $\int G\,dt$ be invariant under parameter transformation. The variation extensor $V^{a(a)}$ is still given by the formulas

$$V^a = \left.\frac{\partial x^a(t,v)}{\partial v}\right|_{v=0}, \quad V^{a'} = \left.\frac{\partial^2 x^a}{\partial t\, \partial v}\right|_{v=0}$$

and if we denote $\int G\,dt$ taken along $x^a = x^a(t,v)$ by I_g then we may conclude from the corresponding development in the preceding problem and the relationship $I_G = K$, that

$$\left.\frac{dI}{dv}\right|_{v=0} = \int_{t_1}^{t_2} F_{;\alpha a}\, V^{a(\alpha)}\, dt;$$

(4.3)

$$0 = \frac{dI_G}{dv} = \left.\frac{dI_G}{dv}\right|_{v=0} = \int_{t_1}^{t_2} G_{;\alpha a}\, V^{a(\alpha)}\, dt.$$

Now since the general method of trying a linear relationship among the primary extensors associated with the integrand functions was successful in one case, it is in order for us to consider it again. Accordingly, we examine

$$AF_{;\alpha a} + F_{a a} + BG_{;\alpha a} + CG_{a} = 0 \tag{4.4}$$

Here we have with some loss in generality limited consideration to the case in which the coefficient of $F_{\alpha a}$ is not zero. We shall also, in order to confine attention to the main case, assume that $F_{;1a}$ and $G_{;1a}$ are not linearly related.

Next, in order to cope with the possibility of over determination caused by the presence of $2N$ equations, we determine the coefficients so that the tensor member $\alpha = 1$ vanishes in the present coordinate system and therefore in all. Thus we obtain the evaluations $A = -1$, $B = -C$ and the equation

$$-F_{;\alpha a} - CG_{;\alpha a} + F_{\alpha a} + CG_{\alpha a} = 0 \tag{4.5}$$

Here we note at once that (4.5) is the Euler equation (4.1) for the simpler problem treated previously except that the integrand function is now $F(x,x') + CG(x,x')$. Thus in essence we have arrived at what is known as the Lagrange multiplier rule for the isoperimetric problem. However, it is necessary for us, because of the heuristic approach to check the compatibility of (4.5) with the requirements of our problem. Accordingly, we now assume that we have given a curve C_0 and a number C such that: (1) C_0 passes through the two given points P_1, P_2 for $t = t_1$, $t = t_2$; (2) meets the requirement $\int_{t_1}^{t_2} G\,dt = K$; (3) sa-

tisfies (4.5) for the given value C, and (4) can be taken as the base curve for an admissible set of comparison curves. We must show that for any admissible set of comparison curves containing the curve C_0 for $v = 0$, dI/dv (at $v = 0$) is zero. To accomplish this, it will suffice to substitute C_0 and the given value for C in (4.5), multiply by $V^{a(a)}$ (the variation extensor corresponding to the set of comparison curves at hand) employ (4.3) and the relationship.

$$\int_{t_1}^{t_2} (F_{aa} + CG_{aa})V^{a(a)}\, dt = \int_{t_1}^{t_2} [(F_{;1a} + CG_{;1a})V^a]'dt = 0$$

The result is

$$\int_{t_1}^{t_2} F_{;aa}\, V^{a(a)}\, dt = 0, \quad \text{or} \quad \left.\frac{dI}{dv}\right|_{v=0} = 0.$$

For a discussion of this problem including the construction of comparison curves, by the classical procedure reference may be made to [1], pages 457-468.

The double integral for $N = 3$. We confine our investigation of the role of extensors in multiple integral problems to the simple but illustrative case of the double integral of a function F of three coordinate variables x^1, x^2, x^3 and their first order partial derivatives with respect to each of two parameters u_1, u_2. To shorten the writing we present merely an outline of the problem. A detailed statement may be found in [1] pages 652 to 671.

Instead of parameterized arcs $x^a = x^a(t)$ defined over a t-internal (t, t_2) and passing through two points P_1 and P_2 for $t = t_1$ and $t = t_2$, respectively; the fundamental geometric entities are now parameterized surfaces $x^a = x^a(u_1, u_2)$ defined over a domain D taken in an auxillary u_1, u_2-plane and consisting of a closed curve $C: u_1 = u_1(t)$, $u_2 = u_2(t)$ plus its interior. The surfaces are all required to pass through the same closed curve L defined by: $x^a = x^a[u_1(t), u_2(t)]$

Despite the increased complexity of the framework of this problem as compared to the single integral case, we shall find that we can take over a good bit of the previous work by employing a simple notational device which will render the similarity between the extensors in the two problems readily apparent. Specifically, we shall indicate partial derivatives with respect to the parameters u_1 and u_2 by means of "*matrix primes*", thus

$$x^{(00)a} = x^{a(00)} = x, \quad x^{(10)a} = x^{a(10)} = \partial x^a / \partial u_1$$
$$x^{(01)a} = x^{a(01)} = \partial x^a / \partial u_2.$$

Similarly, Greek indices will now represent 1-rowed matrices, thus $\alpha = (\alpha_1 \alpha_2)$ with α_1, and α_2 zero or positive integers.

A reexamination of our previous work will show that much of it can be taken over to the present two-parameter case without extensive

modification. For example, the Greek letters α and ρ in the definitions of X sup. αa inf. ρr, X sup. ρr inf. αa will now denote the 1-rowed matrices $(\alpha_1\alpha_2)$ and $(\rho_1\rho_2)$, and the symbol 1 (one) which appears in equations (1.3) may be replaced throughout first with (10) and then with (01). However, the last statement in (1.3) (namely, "if $\rho > \alpha$") must be replaced by *if ρ is partly greater than α*, which means that either $\rho_1 > \alpha_1$ or $\rho_2 > \alpha_2$ or both, since α and ρ now represent the matrices (α_1, α_2) and (ρ_1, ρ_2). Also, the equations (2.1), (2.2) may be taken over unaltered except for the proviso that the repeated Greek indices are now to indicate summation over the set of matrices (00), (10), (01), (11) - if $M = (11)$. With regard to (2.2), it should be observed that if $E_{(11)a} = 0$ then both of $E_{(10)a}$ and $E_{(01)a}$ are tensor ranks. Thus in (3.3) if we take M to be (11) and let F be a function of the x^a, $x^{(10)a}$, $x^{(01)a}$, then $F_{;(11)a}$ is identically zero and the ranks (10) and (01) of the extensor $F_{;\alpha a}$ $(F_{;\alpha a}: F_{;(00)a}, F_{;(10)a}, F_{;(01)a}, 0)$ are now tensor ranks. In order to distinguish extensors of the present type in which the Greek indices range over a set of 1-rowed matrices from their simple counterparts in which the Greek letters draw their values from a set of single numbers 0, 1, ..., M, we call the former *matrix extensors*. For other applications of matrix extensors see [5], [6], [7], [9], [13].

With regard to equation (3.5) the situation is somewhat more complicated. Here we may take the Greek letters to be (00), (10) and replace the symbol 1, which appears explicitly, with the matrix (10) and obtain an extensor, or we may take the range of the Greek letters to be (00), (01) and replace the symbol 1 with (01). Thus we have two extensors here of different ranges. However, an examination of the transformation equations (2.2) with $M = (11)$, will show that we may imbed both of these extensors in the higher range (11) by assigning the missing components the value zero. In order to distinguish these two extensors, we shall denote the one in which differentiation with respect to the first parameter is involved by $F_{\alpha a}$ and the other by $S_{\alpha a}$. Thus we now have three extensors associated with F and the range (00) to (11), namely, $F_{;\alpha a}$, $F_{\alpha a}$, $S_{\alpha a}$. The table of their ranks is

$$
\begin{array}{lllll}
 & (00) & (10) & (01) & (11) \\
F_{;\alpha a}: & F_{;(00)a}, & F_{;(10)a}, & F_{;(01)a}, & 0 \\
F_{\alpha a}: & F_{;(10)a}{}^{(10)}, & F_{;(10)a}, & 0, & 0 \\
S_{\alpha a}: & F_{;(01)a}{}^{(01)}, & 0, & F_{;(01)a}, & 0
\end{array}
\tag{4.6}
$$

Here there is sufficient duplication among the tensor ranks to show that the coefficients B and C in the extensor equation

$$F_{;\alpha a} + BF_{\alpha a} + CS_{\alpha a} = 0$$

can be selected so as to make the tensor ranks vanish and thereby reduce the number of conditions imposed on the N x^{a}'s to N and in addition secure tensor character for the (00) rank. Thus examination of the table (4.6) leads at once to the evaluations $B = -1$, $C = -1$ and we have

(4.7) $$F_{;\alpha a} = F_{\alpha a} + S_{\alpha a}.$$

Comparing this equation with (4.1), we see that $F_{\alpha a} + S_{\alpha a}$ replaces the $F_{\alpha a}$ of the simpler problem.

To verify that a solution of (4.7) will cause dI/dv (at $v = 0$) to vanish, it will perhaps suffice to note the alterations needed to make our previous work applicable to the present case. The variation functions must now be functions of the two parameters u_1, u_2, $V^a(u^1,u^2)$ and must vanish for u_1 and u_2 on C. Thus to adopt (3.9), we must replace t with u_1, u_2. Substitution from the modified form into $\int_D\int F\, du_1 du_2$ now defines $I(v)$. In equation (3.10) we must of course replace the single integral with the double integral over D and replace $F_{;1a}V^{a'}$ with $F_{;(10)a}V^{a(10)} + F_{;(01)a}\quad V^{a(01)}$ and thus the range of α is (00), (10), (01). To obtain the necessary modification of (4.2), we replace $F_{\alpha a}$ in the left member with $F_{\alpha a} + S_{\alpha a}$ and obtain

(4.8)
$$(F_{\alpha a} + S_{\alpha a})V^{a(\alpha)} = (F_{;(10)a}{}^{(10)} + F_{;(01)a}{}^{(01)})V^{a} + F_{;(10)a}V^{a(10)} +$$
$$F_{;(01)a}V^{a(10)} = (F_{;(10)a}V^{a})^{(10)} + (F_{;(01)a}V^{a})^{(01)}.$$

Obviously, the integral part of (4.2) carries over since in evaluating the double integral over D of $(F_{;(10)a}V^{a})^{(10)}$, we may integrate first with respect to u_1 with u_2 held constant and thus get zero because the V^{a} vanish on the boundary C. Similarly in evaluating the double integral of the last term of (4.8) we would integrate first with respect to u_2.

Thus to sum up dI/dv (at $v = 0$) is given by the double integral of the extensor contraction $F_{;\alpha a}\ V^{a(\alpha)}$ and since the extensor $F_{;\alpha a}$ may be replaced by $F_{\alpha a} + S_{\alpha a}$ according to (4.7), we then have

$$\left.\frac{dI}{dv}\right|_{v=0} = \int_D\int (F_{\alpha a} + S_{\alpha a})V^{a(\alpha)}\ du_1 du_2 = 0$$

Finally, we note that the equation (3.8) which is a consequence of the required invariance of $\int F\, dt$ under a parameter transformation has its counterparts in the present two parameter case. The invariance

of $\iint F \, du_1 du_2$ under a parameter transformation implies the relationship

$$F_{;(10)a} x^{(10)a} = F; \qquad F_{;(10)a} x^{(01)a} = 0;$$

$$F_{;(01)a} x^{(01)a} = F; \qquad F_{;(01)a} x^{(10)a} = 0;$$

(see [1] page 665). If we differentiate the first two of these with respect to u_1 and the third and fourth with respect to u_2 and employ the table of values (4.6), we obtain the set of equations

$$F_{\alpha a} x^{(10)a(\alpha)} = F_{;\alpha a} x^{(10)a(\alpha)}, \qquad F_{\alpha a} x^{(01)a(\alpha)} = 0,$$

$$S_{\alpha a} x^{(01)a(\alpha)} = F_{;\alpha a} x^{(01)a(\alpha)}, \qquad S_{\alpha a} x^{(10)a(\alpha)} = 0,$$

with α summed over the set (00), (10), (01), (11). The first and third of these are somewhat analogous to (3.8) and so is the relationship obtained by adding all four equations, namely,

$$(F_{\alpha a} + S_{\alpha a})(x^{(10)a(\alpha)} + x^{(01)a(\alpha)}) = F_{;\alpha a} (x^{(10)a(\alpha)} + x^{(01)a(\alpha)})$$

With regard to the structure of these equations, it should be noted that since $x^{(10)a}$ and $x^{(01)a}$ are contravariant tensors that their derivatives $x^{(10)a(\alpha)}$ and $x^{(01)a(\alpha)}$ are matrix extensors, and hence the last five equations involve invariants in the form of extensor contractions.

* * * * * * * * *

REFERENCES

1. Bolza, Oscar. *Vorlesungen über Variationsrechnung*, Leipzig, Teubner, (1909).
2. Craig, H. V. *On tensors relative to the extended point transformation*, Amer. Jour. of Math. (1937), 764-774.
3. *Vector and Tensor Analysis*, New York, (1943).
4. *On extensors and the Lagrangian equations of motion*, Math. Mag. 26, (1949), 245-251
5. *On multiple parameter Jacobian extensors.* Tensor (New Series) 2 (1952), 27-35.
6. *On Certain linear extensor equations*, Tensor, (New Series) 4 (1954) 40-50.
7. *On certain linear extensor equations: Paper II*, Tensor (New Series) 5 (1955), 77-84.
8. Craig, H. V. and Horton, C. W. *On extensors and the Hamiltonian equations*, Tensor, (New Series) 2 (1951) 47-52.
9. Craig, H. V. and Townsend, B. B. *On certain metric extensors*, Pac. Jour. of Math., 3 (1953) 25-46.
10. Katsurada, Yoshie. *On the parallel displacement of arc*, Tensor (New Series) 2 (1954) 85-88.

11. Kawaguchi, A. *Die Differentialgeometrie höherer Ordnung I. Erweiterte Koordinatentransformationen and Extensoren,* Jour. Fac. Sci. Hokkaido Imp. Univ., 9 (1940) 1-152.

12. *Generalizzazioni del calcolo tensoriale e delle, sue applicazioni.* Randiconti dell' Accademia Nazionale dei Lincei (classe di Scienzefisiche, matematiche e naturali) XV (1953) 251-261.

13. Kawaguchi, Michiaki. *On a generalization of the multiple parameter extensor,* Tensor (New Series) 2 (1952) 85-88.

The University of Texas.

Dear Professor James:

Thanks very much for the beautiful reprints and for your two page letter.

I marvel at your ability in wading thru all your editorial labors. Much of it is due to *this aspect* of author psychology. We see papers in other journals, with this foot note: "Received December 24, 1954, & in revised form July 3, 1955.'' Looking at the first draft of our latest masterpiece and contribution to world culture, we realize it needs revising, but say to ourselves "Even after the trouble of revising it, the editor might want it revised some more, so why not just drop it on him the way it is, and obey his suggestions?" And so, you're overloaded with these chores. What's the solution? If authors read the Magazine carefully enough, they could anticipate your suggested revisions.

If charged a fee, say $3, for every mistake in which they benefit from your editorial experience (no Dr. or lawyer gives his advice away, to anybody who can afford to pay for it) authors would be more careful to save your time and this fee. Most authors, especially beginners, have no conception of the economics of publication. If they had to help you a few days, they'd learn something, and lose some illusions. Just blast away at us in the Magazine explaining the facts of life.

Sincerely yours,

. . . .

Editor's Comment: The solution is not a fee. If editors received adquate *financial* reward for their editorial work, their magazines would be bankrupt very, very soon. The solution is for the author *to put his best* into his papers before submitting it to any magazine. Some authors have their papers read by some one else before final typing; a most commendable practice.

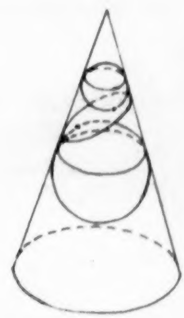
* * * * * * * * * * *

FOCI OF THE CONICS ON A CONE

Robert C. Miller, Jr.

Introduction.

This paper is a discussion of the properties of the space curve that results from considering the locus of the foci of all the conic sections determined by the intersections of a pencil of planes with a right circular cone. In 1882, Dandelin discovered the property that if two spheres are inscribed in a right circular cone and tangent to a plane intersecting the cone, then the points of tangency of the spheres on the plane are the two foci of the conic determined by the intersecting plane [1]. By means of this property the parametric equations of the focal curve are derived. The space curve is then considered in detail as to its general properties and a few of its more interesting special cases.



Dandelin's Property

Coordinates of the Foci. Given any right circular cone, it may be referred to some coordinate system such that the new origin coincides with the vertex of the cone and the axis of the cone lies on one of the coordinate axes. Therefore, we may take the cone

$$(1) \qquad k^2x^2 - y^2 - z^2 = 0$$

as being completely general for this discussion. The equation for a general quadric surface may be written as

$$(2) \qquad Q(x,y,z,) \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{13}xz + 2a_{12}xy + 2a_{23}yz + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0.$$

The sphere

$$(3) \qquad x^2 + y^2 + z^2 - 2fx + g = 0$$

is a special case of (2) with center at $F(f,0,0)$ and a radius $R = \sqrt{f^2 - g}$. It can be shown [2] that the tangent cone from the origin to a non-singular quadric surface $Q(x,y,z)$ is

$$a_{44}Q(x,y,z) - (a_{14}x + a_{24}y + a_{34}z + a_{44})^2 = 0.$$

The tangent cone to (3) is then

(4) $$\left(\frac{f^2}{g} - 1\right)x^2 - y^2 - z^2 = 0$$

We let the cone (1) be the cone (4) and thus

$$g = \frac{f^2}{k^2 + 1}$$

Therefore, the sphere has radius

(5) $$R = \pm f \sqrt{\frac{k^2}{k^2 + 1}} = \pm fh^2$$

using + when $f > 0$ and − when $f < 0$, since R must be > 0. (h^2 has been used to indicate that the positive root must always be taken.)

Let us now consider a general plane in space as being

(6) $$ax + by + cz + d = 0$$

We may consider $d < 0$, since if $d = 0$ the plane passes through the origin and is of no interest. For (6) to be tangent to (3), the distance from F to the plane must equal the radius of (3). The equation for the distance from a point to a plane gives

$$D = \frac{af + d}{\sqrt{a^2 + b^2 + c^2}} = \frac{af + d}{T^2}$$

By (5), and since $R > 0$

$$\pm h^2 f = \left| \frac{af + d}{T^2} \right|$$

where the sign of $h^2 f$ depends on the sign of f as previously explained. Thus we see that when F and 0 are on opposite sides of the plane, i.e., $D > 0$, we have

(7) $$\pm h^2 f = \frac{af + d}{T^2}$$

and when $D < 0$,

$$\pm h^2 f = -\frac{af + d}{T^2}$$

Solving for f in (7) and (8), we have two cases:

I. $f > 0$:

a. $f = \dfrac{d}{h^2T^2 - a}$; F and 0 on opposite sides of the plane.

b. $f = \dfrac{-d}{h^2T^2 + a}$; F and 0 on same side of plane.

II. $f < 0$:

a. $f = \dfrac{-d}{h^2T^2 + a}$; F and 0 on opposite sides of the plane.

b. $f = \dfrac{d}{h^2T^2 - a}$; F and 0 on same side of plane.

From I and II the same solutions for f are obtained, but the opposite situations of the position of F occur. Thus, the two solutions

$$f_1 = \frac{d}{h^2T^2 - a}, \quad f_2 = \frac{-d}{h^2T^2 + a}$$

give the x- coordinates of the centers of the two spheres inscribed in the cone and tangent to the plane. Note that for the part of the cone where $x > 0$, f_1 gives F on the opposite side of the plane from 0, and f_2 gives F on the same side as 0. While for the portion of the cone where $x < 0$, the roles of f_1 and f_2 are reversed.

We proceed to determine the points of tangency of the plane with the two inscribed spheres. It has been shown that f and g of (3) are functions of the coefficients of (1) and (6). Further, it is known that (6) is tangent to each sphere and therefore that the direction cosines of a line perpendicular to the plane are the same as those of a line from F to the point of tangency on the sphere. Hence, the parametric equations of a line on F and perpendicular to (6) are

$$x = f + \frac{as}{T^2}, \quad y = \frac{bs}{T^2}, \quad z = \frac{cs}{T^2}. \tag{9}$$

Substituting these values in (6), solving for s, and substituting this value of s in (9) gives

$$x = f - \frac{ad + a^2f}{T^2}, \quad y = -\frac{bd + abf}{T^2}, \quad z = -\frac{cd + acf}{T^2}$$

Since f may have the two values previously obtained, the points of tangency are

(10)
$$x = \frac{\pm d(T^2 \mp ah^2)}{h^2T^4 \mp aT^2}, \quad y = \frac{-dbh^2}{h^2T^4 \mp aT^2}, \quad z = \frac{-cdh^2}{h^2T^4 \mp aT^2}.$$

The points (10) have been derived as the points of tangency of the plane (6) with two spheres that may be inscribed in the cone (1) and tangent to (6). Thus by Dandelin's property, the points (10) are the foci of the conic section determined by the intersection of the plane (6) and the cone (1).

The Focal Curve. We now proceed to the study of the locus of the foci. Due to the symmetrical character of the cone, the axis of which lies on the x-axis, we may consider a pencil of planes whose axis is parallel to the xz-plane as being completely general. This is true since for any pencil one wishes to consider the axis of the pencil will be parallel to some plane on the x-axis. The axis of the pencil will also form a certain angle with the x-axis. The curve resulting from such a pencil of planes will be congruent to the curve resulting from a pencil whose axis is parallel to the xz-plane and forms the same angle with the x-axis.

A pencil of planes is written $L_1 + mL_2 = 0$ where L_1 and L_2 are two planes and m is the parameter. We choose to write the pencil whose axis is parallel to the xz-plane as

$$a_1x + my + c_1z + d_1 + d_2m = 0$$

Recalling that $h^2 = \sqrt{\dfrac{k^2}{k^2+1}}$, the parametric equations of the focal curve are

(11)

$$x = \frac{\pm (d_1 + d_2m)(T^2 \mp a_1h^2)}{h^2T^4 \mp a_1T^2},$$

$$y = \frac{-mh^2(d_1 + d_2m)}{h^2T^4 \mp a_1T^2},$$

$$z = \frac{-c_1h^2(d_1 + d_2m)}{h^2T^4 \mp a_1T^2},$$

where

$$T^2 = (a_1^2 + m^2 + c_1^2)^{\frac{1}{2}}.$$

The *order* of a space curve is defined as the number of it (real and imaginary) intersections with a general plane. Therefore, we

consider the intersections of (11) with a plane

$$px + qy + rz + s = 0.$$

Substituting (11) gives the result

$$h^2(d_1 + d_2m)(a_1p + mq + c_1r) - h^2s(a_1^2 + m^2 + c_1^2) =$$

$$\pm \sqrt{a_1^2 + m^2 + c_1^2}\,[p(d_1 + d_2m) - a_1s\ ].$$

We may now rationalize this equation and be certain that no extraneous roots will be introduced since the sign of the radical is $\pm$. The result of rationalizing the equation is a quartic equation in m. Therefore, we conclude that the curve is of order four and hence, in general, a quartic curve, C_4.

Let us now determine whether the curve is ever a plane curve, and if so, when. It is known that, if the curve is a plane curve, it must lie in a plane which passes through the origin. Therefore, we consider the intersections of the curve with some plane $px + qy + rz = 0$. Making substitutions, for x, y, z, from (11) gives

(12) $$h^2\ d_1 + d_2m\ \ [a_1p + qm + c_1r] = \pm\ (d_1 + d_2m)p\sqrt{a_1^2 + m^2 + c_1^2}.$$

Now if the curve is a plane curve, it will lie entirely in some plane. For this to be true the coefficients of the plane and the coefficients of the pencil must be such that both sides of the equation above will be *identically zero* in m. This is true if $d_1 = d_2 = 0$, but this is the degenerate case of the curve becoming the origin. Let $d_1 + d_2m \neq 0$. Consider the right side of the equation. Since the term in the radical is the sum of squares, which cannot all be zero, it cannot be zero. Therefore, $p = 0$ must be true. Now from the left side of the equation, we see that, since $p = 0$, $qm + c_1r = 0$ must be true. It follows that $q = 0$ and either $r = 0$, or $c_1 = 0$; but $c_1 = 0$ is the condition that must apply. Thus we see that the only time that the curve is plane is when $c_1 = 0$, that is when the axis of the pencil is perpendicular to the axis of the cone.

For convenience we now choose a particular cone where $k^2 = 1$, and h^2 becomes $1/\sqrt{2}$. This makes no restriction for the general discussion below.

We now determine what type of C_4 the curve is in general. There are two kinds of C_4. A C_4 of the first kind is the basis curve of a pencil of quadric surfaces, while a C_4 of the second kind cannot lie on more than one quadric surface. We shall now show that the C_4 with which we are concerned is of the first kind.

From (11) we obtain

(13)
$$\frac{y}{z} = \frac{m}{c_1}$$
and

(14)
$$-c_1 x + a_1 z = \pm \frac{z}{h^2} \sqrt{a_1^2 + m^2 + c_1^2} \, .$$

Rationalizing (14) gives

$$c_1^2 x^2 - 2a_1 c_1 xz + a_1^2 z^2 = \frac{z^2}{h^4}(a_1^2 + m^2 + c_1^2).$$

Substituting the value of m from (13) and $h^4 = 1/2$ gives

(15)
$$c_1^2 x^2 - 2c_1^2 y^2 - (a_1^2 + 2c_1^2) z^2 - 2a_1 c_1 xz = 0.$$

This equation obviously represents a cone.

We now consider the equation as to whether or not there is some quadric surface other than (15) that carries C_4. The equations of the line on the two foci of a conic section determined by a particular plane are

(16)
$$\frac{x - \dfrac{(d_1 \quad d_2 m)(T^2 - a_1 h^2)}{h^2 T^4 - a_1 T^2}}{\dfrac{-(d_1 + d_2 m)(T^2 + a_1 h^2)}{h^2 T^4 + a_1 T^2} - x} =$$

$$\frac{y + \dfrac{mh^2(d_1 + d_2 m)}{h^2 T^4 - a_1 T^2}}{\dfrac{-mh^2(d_1 + d_2 m)}{h^2 T^4 + a_1 T^2} - y} = \frac{z + \dfrac{c_1 h^2(d_1 + d_2 m)}{h^2 T^4 - a_1 T^2}}{\dfrac{-c_1 h^2(d_1 + d_2 m)}{h^2 T^4 + a_1 T^2} - z} \, .$$

The equation in y and z in (16) gives

$$\frac{y}{z} = \frac{m}{c_1} \, .$$

Now, the equation in x and y in (16) gives

$$ma_1 x + m(d_1 + d_2 m) + y(m^2 + c_1^2) = 0,$$

or

(17)
$$c_1 y^2 + c_1 z^2 + a_1 xz + d_1 z + c_1 d_2 y = 0.$$

The quadric surface (17) must be either a hyperboloid of one sheet or a hyperbolic paraboloid since these are the only two ruled surfaces other than cones. It can be shown [2] that (17) is a hyperboloid of one sheet.

We have now shown that the curve lies on the quadric surfaces (15) and (17). Therefore, in general, the curve is a quartic curve of the

A *rational curve* is defined as a curve which has the property that all its coordinates can be expressed as rational functions of a single variable. It is known that C_4 of the first kind is rational. If we make the substitution $T^2 = \mp (t + m)$ into (11), solve for m in terms of t by rationalizing $T^2 = \mp (t + m)$, and substitute for m in the expressions resulting for the initial substitution, it is easily seen that the curve is indeed a rational curve.

We now proceed to show that the origin (the vertex of the cone) is a double point. It is known that a space quartic curve can have at most one double point. If the origin is a double point, a plane through the origin must have two points of intersection with the curve at 0. Rationalizing (12) gives

$$(d_1 + d_2 m)^2 \ [h^4(a_1 p + qm + c_1 r)^2 - p^2(a_1^2 + m^2 + c_1^2)] = 0.$$

From this we see that $m = (-d_1)/(d_2)$ is a double root. This is the value of m that gives that plane of the pencil that passes through the origin. Therefore, the origin is a double point.

Having shown that a double point exists, we now determine whether the double point is of the cuspidal or the nodal type. The equations of a tangent to a curve at the point (x_1, y_1, z_1) are

$$\frac{x - x_1}{\left(\frac{dx}{dm}\right)_1} = \frac{y - y_1}{\left(\frac{dy}{dm}\right)_1} = \frac{z - z_1}{\left(\frac{dz}{dm}\right)_1}.$$

By taking the derivatives of (11) with respect to m and substituting the value of $m = (-d_1)/(d_2)$ we find the equations of the tangents to the two branches of the curve at the origin to be

$$\frac{x}{\pm d_2 \sqrt{a_1^2 + \frac{d_1^2}{d_2^2} + c_1^2 - a_1 d_2 h^2}} = \frac{y}{d_1 h^2} = \frac{z}{-c_1 d_2 h^2}, \tag{18}$$

We now choose to write the tangents as the intersections of the planes,

$$d_1 h^2 x = \left(d_2 \sqrt{a_1^2 + \frac{d_1^2}{d_2^2} + c_1^2 - a_1 d_2 h^2}\right) y,$$

$$-c_1 d_2 h^2 x = \left(d_2 \sqrt{a_1^2 + \frac{d_1^2}{d_2^2} + c_1^2 - a_1 d_2 h^2}\right) z,$$

and

$$d_1h^2x = \left(-d_2 \sqrt{a_1^2 + \frac{d_1^2}{d_2^2} + c_1^2} - a_1d_2h^2\right) y,$$

$$-c_1d_2h^2x = \left(-d_2 \sqrt{a_1^2 + \frac{d_1^2}{d_2^2} + c_1^2} - a_1d_2h^2\right) z.$$

Case I. $d = 0$, $d \neq 0$. We obtain from the first equations of each set $y = 0$. Examining the second equations of each set, it is obvious that there is no condition that can be set such that

$$d_2 \sqrt{a_1^2 + c_1^2} - a_1d_2h^2 = -d_2 \sqrt{a_1^2 + c_1^2} - a_1d_2h^2,$$

but if we let $c_1 = 0$, the second equations both give $z = 0$. Hence $y = 0$, $z = 0$ is obtained from both sets, and the tangents are coincident. It must, however, be remembered that $c_1 = 0$ is the condition which gives a plane curve.

Case II. $d_1 \neq 0$, $d_2 = 0$. The second equations give

$$-c_1d_2h^2x = (\sqrt{a_1^2\, d_2^2 + d_1^2 + c_1^2\, d_2^2} - a_1d_2h^2)\ z,$$

$$-c_1d_2h^2x = (-\sqrt{a_1^2\, d_2^2 + d_1^2 + c_1^2\, d_2^2} - a_1d_2h^2)\ z.$$

From each of the above equations, we obtain $z = 0$. Considering the first equations, we have

$$d_1h^2x = d_1y,$$

$$d_1h^2x = -\, d_1y.$$

From this result, we see that no condition can be set on the above equations such that they both give the same result since the conditions $d_1 \neq 0$ has already been set.

Case III. $d_1 = 0$, $d_2 = 0$. This gives a pencil whose axis passes through the origin, a degenerate case.

In concluding this section, let us summarize. We have now shown that the locus is a quartic curve of the first kind and hence the basis curve of a pencil of quadric surfaces. The curve is a plane curve when the axis of the pencil is perpendicular to the axis of the cone. Since the tangents at the double point coincide only in a special case of the case when the curve lies in a plane, we conclude, finally, that the locus is a nodal space quartic with the node at the vertex of

the cone. It should be stressed that the locus has been considered in general, and that the curve will be a nodal space quartic unless the case being studied is a special case.

Special Cases. There are three special cases that we shall consider. The first of these will be the case when the axis of the pencil is tangent to the cone. The second will be when the curve becomes a plane curve. The third, will be a case when the locus becomes a finite curve. In all of these special cases the cone $x^2 - y^2 - z^2 = 0$ will be the right circular cone that is used.

In the first case we choose to write the pencil of planes as

$$a_1x + c_1z + d_1 + m(x + y) = 0.$$

From the parametric equations of the curve one can show that the curve lies on the two quadric surfaces

$$c_1x^2 - 3c_1^2y^2 - (a_1^2 + 2c_1^2)z^2 - 2a_1c_1yz - 2a_1c_1xz - 2c_1^2xy = 0$$

and

$$c_1y^2 + c_1z^2 + a_1xz + c_1xy + d_1z = 0.$$

By using the theorem [2] below we can show that the space quartic degenerates into a space cubic and a generator of the cone.

Theorem: The line

$$x = A + \lambda s, \quad y = B + \mu s, \quad z = C + \nu s$$

will lie entirely on the quadric surface Q, if and only if, $L_1 = L_2 = L_3 = 0$, where

$$L_1 = ½[\lambda Q_1(A,B,C) + \mu Q_2(A,B,C) + \nu Q_3(A,B,C)],$$

$$L_2 = Q(A,B,C), \text{ and } \quad L_0 = q(\lambda,\mu,\nu)$$

with

$$Q_1 = \frac{\partial Q}{\partial x}, \quad Q_2 = \frac{\partial Q}{\partial y}, \quad Q_3 = \frac{\partial Q}{\partial z},$$

and

$$q = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{13}xz + 2a_{12}xy.$$

The line the two quadric surfaces have in common must lie on the origin and the point of tangency. By using the parametric equations of this line and applying the above theorem it is easily shown that the line is common to both quadric surfaces. This means that the quartic curve degenerates into a straight line and a space cubic. It should be noted that the only condition set was that the axis of the pencil be tangent to the cone, and that the choice of the coefficients a, c, d has no effect other than to give a particular C_3 Note also that if the axis of the pencil is a generator of the cone the locus is degenerate.

By the means of several figures we consider the locus when it is a plane curve. In Fig. 1, the point P is the point of intersection of the plane and the axis of the pencil. In this case there are two double points: one at the origin and one on the x-axis at $x = d_1$. Both branches of the curve go to infinity, in different directions, when the determining plane becomes parallel to a generator of the cone. Notice that the two branches are cubic curves, but that the two branches make up the quartic curve that is obtained algebraically.

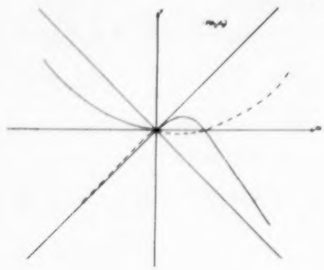


Figure 1

Figure 2 shows the only case when the double point at the origin is of the cuspidal type. It was shown previously that this was the case when the axis of the pencil lies on the y-axis. Here the symmetry of the two branches is very striking.

Figure 2

Fig. 3 shows the case when the axis of the pencil is tangent to the cone. The line is indicated by the dashed boundary line of the cone.

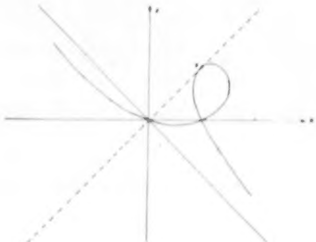


Figure 3.

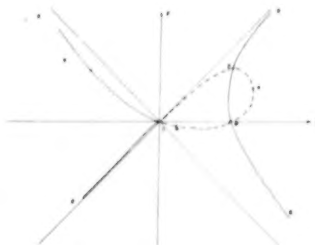


Figure 4.

Fig. 4 gives the picture of the curve when P lies within the cone. The foci on the section of the curve AO are associated with the foci on the section FO. The section AB is associated with the section DB. BC is associated with BE, CP with G, and PO with GP.

The third case is when the axis of the pencil is parallel to the axis of the cone. The parametric equation of such a pencil is

$$y + mz + md_1 = 0.$$

From the parametric equations of the quartic curve one can show that the curve lies on the two quadric surfaces

$$h^4x^2 + d_1 z = 0,$$

$$y^2 + z^2 + d_1 z = 0.$$

The first surface represents a parabolic cylinder whose generators are parallel to y-axis. The second represents a circular cylinder whose generators are parallel to the x-axis. The orientation and the type of the surfaces make it clear that the curve is finite.

REFERENCES

[1] R. Courant and H. Robbins, *What is Mathematics?* New York, 1941, pp. 199-201

[2] A. Dresden, *Solid Analytical Geometry and Determinants.* New York, 1930, pp. 160, 168, 230.

U.S. Naval Ordnance Test Station, China Lake, Calif.

Dear Mrs. James:

Am enclosing my check for the annual donation to the cause of the Magazine. All signs indicate a remarkable work being done by Mr. James and his staff. I was glad to see the list of countries published, into which the journal goes.

With all my best wishes for continued success, I am

Yours very sincerely,

S.T. Sanders

(Prof. Sanders founded the National Mathematics Magazine, the predecessor of Mathematics Magazine. Ed.)

GEOMETRIC REINFORCEMENT

R. Lariviere

Some readers may recall that, though as students or review students they became well satisfied with the explanations given them of the rules for adding or subtracting signed numbers, the explanations of the rules for their multiplication remained fairly abstract. The following geometric illustration, which involves only simple equations and similar triangles, may satisfy the need for some concrete visualization of the process.

Let OH and OK be two intersecting number axes with positive directions indicated by arrows. Let OA be unit length in the positive direction on the first axis OH. Let OB represent the multiplicand M, in magnitude and sign on the same axis as OA. Let OC represent the multiplier m, in magnitude and sign, on the second axis OK. Join AC. Through B draw a parallel to AC to meet OK in D. Then OD represents the product of m and M since, from the similar triangles OAC, OBD, $OD = OC \cdot OB/OA$ and $OA = +1$.

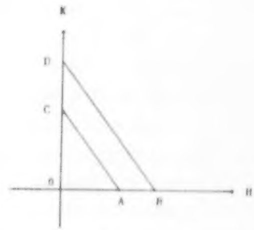


Fig. 1

This result is equally true in figures (2), (3), and (4).

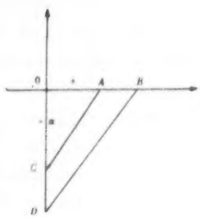


Fig. 2

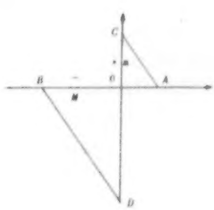


Fig. 3

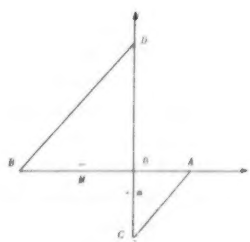


Fig. 4

The algebraic rule that the sign of a product of two factors of unlike sign is negative is geometrically corroborated in figures (2) and (3) where m and M have unlike signs and the product OD is negative, while the rule that the sign of a product of two factors of like sign is positive is observed to hold in figure (1), and also in figure (4) where both factors are negative.

Figure (4) is often found more convincing than most algebraic proofs of the fact that the product of two negative factors is positive.

University of Illnois in Chicago

DIG THAT MATH

Wilfrid Dellquest

QUEEN OF SCIENCES: Everybody knows that Mathematics is Queen of Sciences and now that Occidental College is staging a Mathematical Field Day, with exciting hassles and brannigans, I figure that math may at last be coming into its own as a popular national sport. With proper promotion, math may evoke squeals like Elvis Presley and sharp operators may be scalping tickets to math field days at 35 bucks a ducat...Giving the subject a long, cool look, it is evident that math has suffered in three ways; it has no competitive spirit; it has no crime interest and it has no sex appeal.

We must spice up math and give it a jigger of yell-juice by holding mathematical tugs of war between big league teams in coliseums and bowls...It could be breathtaking. You may some day read a press notice like: " Seventy thousand cheering fans saw the Jolly Geometricians trounce the Trigonometry Tigers in a close season opener at Einstein Bowl. The Geomets came prancing upon the field with axioms to grind and during the warm-up period made practice passes and propositions at the cheer leaders all of whom had stunning figures. The Oxy rooting section was filled with cube rooters rehearsing the team yell: " X plus Y! X plus Y! Hold that factor! Racketty Rax-Univacs! O-oo-x-yy! " Multiplicandy venders climbed the aisles selling Euclid bars and square rootbeers while using their mathematical knowledge to short change the customers. The score was tied X to Y at the end of the half. During the third quarter, the Trigs were penalized X yards for holding a decimal point and the Geomets were sent back to the Y yard line for whistling at a cute angle. In an exciting final period, the Geomets fumbled the ball when they fell on their hypotenuses. The

(continued on page 221)

A SIMPLE EXTENSION OF THE ARISTOTELIAN DEFINITION OF MATHEMATICS

(Non-technical)

H. Kennedy

The great difficulty in getting even two mathematicians to agree upon a definition of mathematics would seem to lie in the fact that a definition must come from without rather than from within mathematics. Thus the people who talk mathematics have difficulty talking about mathematics. Most mathematicians, however, will agree that their subject has its origins in the consideration of external reality. Consider, for example, the counting numbers. Whether one takes these as basic to mathematics, as the Intuitionists do, or whether one defines them with the terms of logic, as the Logicists do, certainly all agree that there would be no counting going on unless there were things to be counted.

This much is in agreement with Aristotle, for he says,[1] "The mathematicians's investigations are about things reached by abstraction (τὰ εξ αφαιρέσεως); for he investigates things after first eliminating all sensible qualities, such as weight, lightness, hardness and its contrary, also heat and cold and all the other sensible contrarieties, leaving only the quantitative and continuous ... and the properties of these things *qua* quantitative and continuous ...". But Aristotle would seem to confine mathematics to the study of quantity and continuity, or at least to a property of matter that can be arrived at by the process of abstraction he describes, for he says,[2] "When we think of mathematical objects we conceive them, though not in fact separate from matter, as though they were separate."

It is this restriction that modern mathematicians object to, since in part of their study they are concerned with relation *qua* relation and not as the relations of quantity or continuity, for relation as an object of study does not arise from the process of abstraction that Aristotle describes: relation is not a property of matter in the sense that quantity and continuity are. We must, then, extend Aristotle's definition if we are to include in it this kind of activity. Following a suggestion [3] of Henry Veatch let us broaden the domain of mathematics, as the Aristotelian conceives it, to include matter drawn from the category of relation as well as from that of quantity. By this simple extension we bring the activity of modern mathematicians within our definition.

But is not the germ of this extension already in the thought of Aristotle? For consider that his purpose was not so much to tell mathematicians what to do as to describe what they were actually doing, and we see that whereas he arrived at mathematical entities by a process of abstraction from the sensible qualities of a substance, nevertheless the end result was not substance quantified but the "properties of these things *qua* quantitative and continuous." Aristotle contrasts[4] a

unit and a point by saying the former is a "substance without position" (οὐσία ἄθετος) and the latter is a "substance having position" (οὐσία θετός). since, as we have seen, mathematical objects are for Aristotle not quantified substance but the properties of quantity and continuity, how can we explain his use of the word "substance"? W. D. Ross comments:[5] "Aristotle's use of the term οὐσία in defining the unit and the point is not strictly justified, since according to him mathematical entities have no existence independent of subjects to which they attach. But he can call them οὐσίαι in a secondary sense, since in mathematics they are regarded not as attributes of substances but as subjects of further attributes." It would seem that Aristotle did not hesitate to consider the result of his first abstraction to be like his starting point(*like* in being a subject of further attributes) and, it appears, fit subject for a similar process of abstration. It is by exercise of this second abstraction that we arrive at the study of relation *qua* relation.

Our extension of the Aristotelian definition appears to be the natural one and is, at any rate, the one that seems to be indicated by Aristotle himself.

FOOTNOTES

1. *Metaph.* K.3. 1061^a28-$^b3$.
2. *De An.* III.7. 431^b12-16.
3. *Philosophy and Phenomenological Research*, Vol XI, No.3(1951)p.350n.
4. *An. Post* 87^a35-7.
5. *Aristotle's Prior and Posterior Analytics with Introduction and Commentary by W. D. Ross* (Oxford at the Clarendon Press, 1949) p.596.

University of Florida

CURRENT PAPERS AND BOOKS

Edited by
H. V. Craig

This department will present comments on papers previously published in the Mathematics Magazine, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to *H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.*

* * * * * * * * *

COMMENTS ON "LONG-SHORT LINES"

Vol. 30, No. 1

COMMENTS ON GLEN JAMES' "LONG-SHORT LINES"

by R.W. Bagley, P.O. Bell, and J.D. McKnight.

When an article begins with a statement such as, "It is not what mathematicians say but the way they say it that makes mathematics almost un-understandable to the layman, the student and the mathematician who is not an expert in the subject matter at hand", one is thereby forewarned that very little if any mathematics is to follow. Such anti-mathematics propaganda as is exhibited in the article we discuss here is extremely harmful in that it certainly misleads students, causing good students to retreat and poor students to decide mathematics is just another education course, hence not so bad after all. Of course, there are those who will not agree that this is harmful to mathematics.

The author of the article in question criticizes mathematicians for not using plain English and proceeds to talk of a sequence of sets of curves having as its limit a curve. Actually, since each term in his sequence fails to contain any curve contained in any other term of the sequence the limit is the null set.

The author assumes that he has defined "pseudo-lines". According to his definition, just what the class of pseudo-lines is depends on the eyesight and pencil of some unnamed person. Usually mathematical definitions do not depend on such things. Of course, the author may have a similar definition of "plain English".

* * * *

REGARDING THE COMMENT ON MY PAPER, by Glenn James

The first sentence is taken out of context. The very next sentence in my paper says that the language of mathematics is essential to mathematics. This should make it clear that by "plain English" I mean non-

technical language and that I am not defaming the technical language that we use and must use to advance our subject. But there is something deeper than that involved here.

In this paper and in other semi-popular and popular papers that we publish, we are assuming that mathematics is a system of thinking independent of the particular language that expresses it (except in so far as its concepts become linked to certain symbols by continued association). For instance, whether calculus is written in English, French, German, Russian, or Chinese it is essentially the same, but one would hardly expect a person who could read only English to understand calculus written in Chinese.

"Good students" are inspired and "poor ones" saved by first showing them the meaning of mathematics in terms of a language which is familiar to them and, or by illustrating with special cases within their experiences. The paper under discussion does both. (Personally, I feel that no matter how much the methods of two men attempting to promote mathematics differ, there is nothing gained by one accusing the other of "anti-mathematics propaganda". "There are many different roads to Heaven.")

The second paragraph in the comment seems to be based on a misreading of my paper. Each set of the several sets of curves discussed, stairsteps, semi-circles, cycloides etc., is a one-parameter family of curves which approaches a certain curve (line) as the parameter approaches a certain limit while the lengths of segments of the curves do not approach the length of the corresponding segments of the limit curve. The reference to Tonelli would clarify this point for those who desire a more rigorous treatment than would be fitting in the present paper.

"Pseudo-curves" are defined by the indicated sequences in the same logical sense that a certain sequence defines the squareroot of two. Speaking of going out a sequence until the curves "appear to be straight lines" is analogous to speaking of a "high degree of accuracy" in an approximation to the squareroot of two. Both can be stated more precisely in terms of decimals and normals but that would be inappropriate in a popular paper.

* * * * * * *

SOME REMARKS CONCERNING A COMMENT by Homer V. Craig

Regrettably it sometimes happens that writers who are capable of excellent scholarship and habitually employ due care in the preparation of their own papers become rather hasty and inattentive when they decide it is their duty to expose the evil in the work of someone else. I must confess that I have been irritated by a few sporadic instances in point and as a consequence I may have failed to read *Comments on 'Glen' James Long-Short Lines* with sufficient sympathy for good understanding. However this may be, it is my conviction at present that there is surely misinterpretation involved in this adverse criticism. In the following

discussion I shall refer to the paper itself as LSL and denote the comment by C.

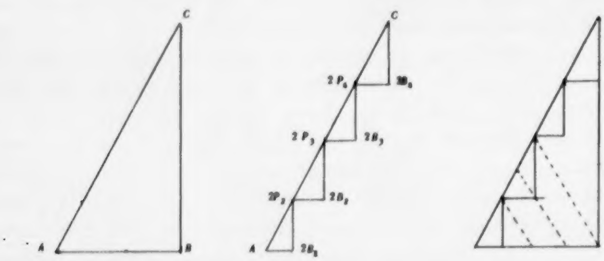
The first two sentences of LSL are: "It is not what mathematicians say but the way they say it that makes mathematics almost un-understandable to the layman, the student and the mathematician who is not an expert in the subject matter at hand. Thus the highly concentrated ideosyncratic language of mathematics is a handicap as well as an essential aid in the development of the subject". With regard to these statements it is certainly true that laymen, students, etc., have complained that highly technical articles in mathematics are hard to understand. The second sentence by virtue of the use of the word essential implies that there is no satisfactory substitute for the technical language as far as the development of the subject is concerned. Thus examination will show that there is no adverse criticism in LSL of mathematicians for not employing plain English exclusively, but rather the expression of (1) the observation that the uninitiated do find highly technical articles in mathematics difficult to understand; and (2) the thought that preliminary discussions for the layman of mathematical topics in language familiar to the layman might serve a useful purpose. I assume that there is general agreement on the fact that now more than ever before there is an urgent need for a more wide spread appreciation of the power of mathematics and so the whole matter of expository techniques for articles addressed to the layman is crucial. Certainly, the proposal to describe mathematical situations in ordinary language does not call for the accusation that the author of LSL (and perhaps by implication the journals cited in LSL) are guilty of anti-mathematics propaganda. The first sentence in LSL is of course somewhat ideomatic and if one were to stop reading temporarily just before the word "makes" and mentally supply the word "matters" the entire sentence would have a bad flavor, but to assert that the first sentence forewarns the reader that little if any mathematics is to follow seems to the writer to be somewhat akin to the adoption of an unalterable program calling for a fair trial in the morning and a hanging in the afternoon.

In some cases at least technical language can be translated into plain English without loss of precision as witness the expression "the limit is the null set" which might be rendered: does not have a limit. The dictionaries that I have consulted seem to indicate that the ordinary sense of the noun set is as given in the Oxford Dictionary:
"A number of particular things that are united in the formation of a whole . Thus the term "null set" regardless of its value for the specialist might very well strike the layman as being somewhat puzzling or perhaps even contradictory.

Certainly the goals of the interested layman who wants to get a glimpse of some of the things that mathematicians talk about are quite different from those of the research specialist; and an article address-

ed to the layman with the understanding that the strict technicalities are to be replaced by descriptions in familiar terms should be interpreted with this in mind. It seems to me particullarly from this viewpoint that the objection raised in C with regard to the existence of the limits of the sequences of curves introduced in LSL is fanciful rather than real.

In order to examine this situation in detail, we replace figure (1) of LSL with a set of figures.



Here in these figures we have certain triangles which for simplicity we shall assume to be right angled with equal legs. The first term of the stair sequence to be considered is the pair of sides *AB*, *BC* "united in the formation of a whole" namely the broken line or stair element *ABC*. The second term of the sequence is obtained by the following procedure: (1) the original hypotenuse is divided into four equal parts (we denote the end points of these pieces by $2P_1$ (or A), $2P_2$, $2P_3$, $2P_4$, $2P_5$ or C with the index 2 before the letter P used to correlate these points with the second term of the sequence of stair lines), (2) on each of the parts $A\ 2P_2$, $2P_2\ 2P_3$, etc., as an hypotenuse there is constructed by two line segments one parallel to *AB* and one to *BC* a right triangle, (3) these added segments "united in the formation of a whole" constitute the third term of the sequence. Thus the second term is a stairline made up of four tread lines and four risers. This process is to be continued without end and so we have under consider-tion an infinite sequence of stair lines.

Now that the sequence in question is defined, what is meant by saying that *AC* is the limit of the sequence? Here we are obviously concerned with distances from the points of the various stair lines to *AC* and so a natural elementary approach would be to introduce the set of perpendiculars to *AC*. Each of these perpendiculars will intersect each stairline in a unique point. The point of intersection of a given perpendicular with the first term *ABC* of the stair line sequence might be designated as intersection point number one associated with that particular perpendicular. Similarly, the point of intersection of this

same perpendicular with term number two of the stair line sequence namely the stair line $A\ 2B_1\ 2P_2\ 2B_2\ \ldots\ C$ may be selected as point number two etc. Thus associated with each perpendicular to AC there is determined by this process an infinite sequence of points and likewise the distances of these points (associated with a given perpendicular) from AC constitute an infinite sequence of non-negative numbers say $d_1, d_2, d_3, \ldots,$ associated with the perpendicular selected. Now if it should be that, no matter what perpendicular of our infinite collection of perpendiculars is chosen, that the associated sequence of distances $d_1, d_2, d_3, \ldots,$ has limit zero; would it not be fitting and proper to say that AC is the limit of the sequence of stair lines?

This last question of course raises another question: What is meant by the statement, the limit of $d_1, d_2, d_3, \ldots$ is zero? Using the fundamental notions "less than" and "greater than", this may be formulated as follows. The limit of a sequence $d_1, d_2, d_3, \ldots,$ of non-negative numbers is zero means that given any positive number (which we shall call the test number) there exists an associated positive integer m having the property that if the first m terms of the sequence are discarded, then each of the remaining terms of the sequence will be less than the test number, or briefly, given $\epsilon > 0$ there exists an associated integer m such that if $n > m$ then $d_n < \epsilon$.

In order to illustrate this particular limit concept and at the same time proceed with the main question let us examine in detail the sequences associated with the perpendiculars to AC, simplifying the computation by taking AC, to be two units in length. As the first sequence to be studied, we select the one associated with the perpendicular to the midpoint of AC namely the line $2P_3B$. Now since the length of $2P_3B$ is one and since $2P_3$ is on AC, the first two terms of our sequence are 1, 0. But the process of constructing the stair lines retains the preceding division points, i.e., $2P_3$ is also a point of the third stair-line ($2P_3$ is $3P_9$) and of every succeeding stair line. Thus the distance sequence associated with the midpoint perpendicular is 1, 0, 0, ..., ; specifically the first term is one and each of the following terms is zero. In this particular case for any given number $\epsilon > 0$, we can take $m = 1$ and assert that each term after the first is less than ϵ. Evidently the first term of this sequence is greater than the first term of any other distance sequence.

Next let us examine one of the sequences having maximum second term, say the one belonging to the perpendicular through $2B_2$. The first term of this sequence is ¾, the second ¼ and the remaining terms are zero, i.e., the sequence in question is ¾, ¼, 0, 0, 0, ..., . In this case if an $\epsilon > ¼$ is given, we could take $m = 1$, while if $0 < \epsilon < ¼$, then we could take $m = 2$ or of course any larger number. Similarly, the maximum value for the third terms of the various sequences is $(¼)^2$. We now construct a sequence by selecting the maximum first term, the maximum

second term, etc.,from our infinite collection of sequences, thus 1, $(\frac{1}{4})$, $(\frac{1}{4})^2$, $(\frac{1}{4})^3$, $(\frac{1}{4})^4$, ..., $(\frac{1}{4})^n$, ... Evidently, the limit of this sequence is zero, for $(\frac{1}{4})^2 < 1/10$ and for n a positive integer $(\frac{1}{4}^2)^n < (1/10)^n$. Now suppose that we are given a positive number ϵ in decimal form, we can obviously choose n so that $(1/10)^n < \epsilon$ and hence $(\frac{1}{4})^{2n} < \epsilon$. Thus given an $\epsilon > 0$ we can take m to be twice the number as just determined and this m will satisfy the condition of the definition for the given ϵ and any one of the distance sequences of our infinite collection. Hence it develops that the line AC is the limit, in a simple and natural sense, of the infinite sequence of stair lines under consideration.

The pseudo line corresponding to the sequence of stair lines just discussed is the line AC together with the number $2\sqrt{2}$. Thus a pseudo line is a couplet consisting of an arc and a real number with which couplet there is associated a certain sequence of sets of arcs. In certain topics in theoretical mechanics use is made of the concept particle (a point with an associated number, the mass) and similarly of a line endowed with a number called the linear density of mass. Evidently, if we associate with each of the stair lines a unit linear density then the associated pseudo line may be interpreted as the line AC together with the linear density $2\sqrt{2}$.

Of course the mathematical entities; points, lines, real numbers etc. belong to the realm of ideas rather than to the domain of physical objects. But the introduction of illustrative objects (pictures made with paper and ink etc.) is a universal practice customarily done without apology or explanation of status and does not carry with it the implication that these physical objects are the only entities under discussion. Consequently, while there is some justification for the statement in C that the concept pseudo line depends on a persons eyesight, this of course applies to the illustrative physical objects - the "lines" as drawings on a paper and not to their entirely different counterparts *lines* in the realm of ideas. The infinite process introduced in LSL of course has to do with geometric entities as abstract ideas characterized by certain postulates and does not carry over to the universe of physical objects. In the physical case one cannot repeat the stair line process time after time without ultimately reaching a stage when it is debatable whether or not the process can be continued another time, and this ultimate stage will be contingent on various things such as the individual's eyesight.

However, there is obviously an oversight involved in LSL and a constructive suggestion calling for the missing explicit definition of pseudo line in the abstract seems to be in order. Incidentally isn't it somewhat of a characteristic of mathematics that criticism can usually be constructive?

COMMENTS ON "A CERTAIN PROBLEM IN MECHANICS"

(by M. R. Spiegel, Vol. 30, No. 2, Nov.-Dec. 1956.)

COMMENT BY K. L. Cappel

The apparent "paradox" presented in Mr. Murray R. Spiegel's note appears to me to lie in the mistaken treatment of equation (5) as containing only one unknown, x, after setting v equal to zero. Up to this point, the formal solution of the problem is perfectly correct, and may be continued as follows; as an ordinary *differential* equation:

$$v = \frac{dx}{dt} = 4 - 2x. \qquad \text{On reparatirey variables:}$$

$$dt = \frac{dx}{4-2x}, \quad \therefore \quad t = -\tfrac{1}{2}\, lu\,(4-2x) + C'$$

$$x = 0 \text{ when } t = 0, \quad \therefore \quad \tfrac{1}{2}\, lu4$$

$$\therefore \quad t = -\tfrac{1}{2}\, lu\, \frac{4-2x}{4}$$

$$x = 2(1 - e^{-2t})$$

which when differentiated twice will be found identical with equation (1) of the article.

In other words, a differential equation cannot be solved by setting the derivative equal to zero as one of the boundary conditions, without a further integration.

The Franklin Institute

* * * * * * * * *

COMMENT BY William E. F. Appuhn

I. In physical applications of mathematics we regularly have to restrict ourselves to certain ranges of values of the variable in the expressions and equations which we are using, to comply with the restrictions of the physical problem.

II. The first equation of (3) is true for all positive values of v, for finite time, in the application i.e. $0 \leq v \leq 4$. The second equation of (3), i.e. $\frac{dv}{dx} = -2$, is deducible from $v\,\frac{dv}{dx} = -2v$, only for $v \neq 0$, and hence only for finite time. (Division by zero not being permissible).

III. If we restrict ourselves to finite time, then $v = 4 - 2x$, and $0 \leq x \leq 2$, since by integration of $v = 4e^{-2t}$, using initial conditions, we have $x = 2 - 2e^{-2t}$. From this equation we note further

that $x \to 2$ as $t \to \infty$ and then $v \to 0$. Since $x < 2$ for all finite time we are not justified in letting $x = 2$, and getting $v = 0$. The statement in the article to the effect that the object never comes to rest, assumes time to be finite, in which case $x < 2$ and hence $v = 4 - 2x$ is not zero, but is > 0, and there is no inconsistency.

St. John's University

* * * * * * * * *

COMMENT BY D. A. Breault

We are given that

$$\frac{dv}{dt} = -2v \tag{1}$$

and that

$$v = 4 \text{ and } x = 0 \text{ when } t = 0. \tag{2}$$

By integrating (1) in two different ways, subject of course to the conditions (2), we obtain

$$v = 4 - 2x \tag{3}$$

and

$$v = 4e^{-2t} \tag{4}$$

To show that these results are not contradictory, we integrate (4) to arrive at

$$x = 2 - 2e^{-2t} \tag{5}$$

Now, the true situation is quite clear. Equation (3) states that $v = 0$ when $x = 2$, but (4) and (5) show that the conditions of (3) are approached asymptotically, but never attained. The solution of (3) is misleading in that it implies that x can take on the value 2, at which time v will of course be zero. (4) and (5), however, show that such is not the case.

Carnegie Institute of Technology.

* * * * * * *

COMMENT BY Randall B. Conkling

This seems to be a situation in which Zeno's point is well taken. The system of first order equations corresponding to $\frac{dv}{dt} = -2v$ is

$$(1) \quad \begin{cases} \dfrac{dx}{dt} = v \\[2mm] \dfrac{dv}{dt} = -2v\,. \end{cases}$$

In the xv (phase) plane, the phase portrait of this system is the family of straight lines with slope -2. For the initial conditions $x = 0$, $v = 4$, the representative point starts at (0,4), travels down the line with this v-intercept, but is reluctant to reach the x-axis (here playing the part of Zeno's wall) and does not do so in finite time, or, as Spiegel says, the object never comes to rest.

The parametric equations of the motion are

$$(2) \qquad \begin{cases} x = -2e^{-2t} + 2 \\ v = 4e^{-2t} \qquad t \geq 0. \end{cases}$$

The function defined by these equations is not the same as that defined by $v = 4 - 2x$. Considered as a set of ordered pairs (x,v), the equations (2) yield only pairs for which both members are positive. The question "What is the first member of a pair whose second member is zero?" has no meaning.

The textbook solution becomes questionable at the step in which both sides of the equation $v \dfrac{dv}{dx} = -2v$ are divided by v. This removes all the singularities of the system (1). The whole x-axis in the xv-plane is singular, each point thereon being an equilibrium point of the system. This means, of course, that if the object is placed anywhere on its line of motion, with zero velocity, it will not move. For any other initial conditions, including those involving negative initial velocity, and therefore the lower half of the xv-plane, the representative point moves on a phase line toward the x-axis but will not reach it in finite time.

The textbook question is equivalent to the following: A four gram mass of radioactive material decays at a rate numerically equal to twice its size. What will be its color when it disapears?

New Mexico College of A and MA.
State College, N.M.

* * * * * * * * *

Figures Can Be Fun – Or – The Lighter Side Of Mathematics, by A. H. Hunter. Oxford University Press. 114 Fifth Ave. New York, N.Y.

This book is made up of a series of 150 mathematical problems for adult layman, cast in the form of entertaining anecdotes. When was the City Hall clock right? How old was Humpty Dumpty when he fell? What was the number of Professor Brayne's car? Did King Dormon of Kalota die in the year of Karan or the year of Kelkis? How many muffins did Milly Miffen make? There are hours of entertainment in this book for the mathematically minded, whether their training stopped at

Grade 12 or with a Ph.D.

Irrational Numbers. By Ivan Niven. John Wiley & Sons, 440 Fourth Ave. 167 pages. Aug. 1956. $3.00

Collects material on irrational numbers which is scattered in the literature. Discusses fully the irrationality of the numbers of elementary analysis. Gives a complete treatment of recent work on normal numbers. Offers a complete, detailed exposition of the basic theorems on transcendental numbers. Presents the analytical proof by Hermann Weyl of the uniform distribution theorem on the multiples of an irrational number.

An Introduction to Tensor Analysis. By Leonard L. Barrett. The National Press, Palo Alto, California. October 10, 1956. $2.00

This treatise is an understandable exposition of the principles of tensor analysis. It will enable the mathematician to make use of this mathematical tool in classical mechanics, quantum mechanics, elasticity, electromagnetics, and aeronautics. It is useful to the understanding of such works as *Electromagnetic Theory*, by J. A. Stratton and *The Meaning of Relativity*, by Albert Einstein. A sequel to *Engineering Applications of Vector Analysis*, it presupposes a working knowledge of vector analysis.

The Enjoyment of Mathematics. By Hans Rademacher and Otto Toeplitz translated by H. Zuckerman. Princeton University Press, Feb. 1957 $4.50.

Extract from the Introduction: "Mathematics, because of its language and notation and its odd-looking special symbols, is closed off from the surrounding world as by a high wall. What goes on behind that wall is, for the most part, a secret to the layman. He thinks of dull uninspiring numbers, of a lifeless mechanism which functions according to inescapable necessity. On the other hand, that wall very often limits the view of him who stays within. He is prone to measure all mathematical things with a special yardstick and he prides himself that nothing profane shall enter his realm."

Requiring no more mathematical background than most people acquire in high school (plane geometry and elementary algebra), this book introduces the reader to some of the fundamental ideas of mathematics, the ideas that make mathematics exciting and interesting.

The amateur will savor this book, which has already become a classic in German. Teachers will use it as a touchstone for mathematical talent in their students, and as a means of interesting students in basic mathematical problems. Sophisticated mathematicians will enjoy seeing

again how mathematics can be both simple and deep. This book provides the means for the reader to enjoy mathematics in the same way that he can enjoy music.

A DEVELOPMENT OF ASSOCIATIVE ALGEBRA AND AN ALGEBRAIC THEORY OF NUMBERS, IV, - Errata Vol. 30, No. 1

Page 2: Line 3, read "$E_2[x]$" in lieu of "$[E_2]x$. Line 7, read "$\cong$" for the last two equivalent signs, which are not clearly printed. Line 2 from bottom, read "$\underline{(}$ and $\underline{)}$" in lieu of "(and)".

Page 3: Line 12, read "$\underline{(}$ and $\underline{)}$" in lieu of "(and)". Line 13, read "(,)" in lieu of "(,)". Line 23, delete the word "three".

Page 4: Just below line 22, start a new paragraph: "We give without proof the following: ".

Page 5: Line 1, read "$(A \times B) \times D \cong$" in lieu of "$(A\ B) \times D \cong$ "." Line 9, read "The" in lieu of "Two". Same line, read "semi-ring" in lieu of "semi-rings".
Same line, read "is said to be homomorphic to the semi-ring S_2" in lieu of "and S_2 are said to be homomorphic". Line 17, read "an" in lieu of "and". Line 21, read "(II)" in lieu of "(III)". Line 27, read "C_i" in lieu of "C ."

Page 6: Line 4, delete the parenthesis mark *before* the word "approximately". Line 6, insert at the beginning of this line "$j >$". Same line, delete "for if $j > i > 1$". Same line, insert the word "since" before the word "it". Line 7, read "$\cong$" in lieu "$\ncong$". Line 10, read "forms" in lieu of "form". Line 6 from bottom, insert "ϵ" before the last "I".

Page 7: Line 17, delete entire line and "C_1." at beginning of line 18. Line 21, add the following: "Furthermore if $L = C_i$ and $'_1 = C_i$, we postulate that $L = L_1$. Of course each above replacement element of S must be contained in the coset of L which it replaces. Line 14 from bottom, read "s_1, s_2, s_3" in lieu of "s, s, s".

THE AMERICAN ARITHMETIC, *by John F. Stoddard, A.M.,*

New York, Sheldon Company. Entered according to an act of Congress, in the year 1849, by John F. Stoddard in the Clerk's office of the District Court for the Southern District of New York.

Extract from "Preface"

" Neither a desire of pecuniary gain, nor a wish to appear as an author, prompted the presentation of this work to the public. Having felt the necessity of a more extended and systematic Intellectual Arithmetic for the younger, as well as more advanced pupils, I prepared and used in manuscript, in my own school, for a number of years, such a series of questions as I deemed best adapted to the purpose. After observing the superior mental training derived from their use, and the ease with which pupils thus trained comprehended the more advanced branches of mathematics, I venture to submit them to the public in the following pages, hoping that they may prove as useful to other schools as they have to my own."

Extract from "Suggestions to Teachers."

"For the benefit of those whose experience is limited, I make the following suggestions in regard to the most approved methods of teaching this important branch of study:

The lesson should be assigned previous to recitation, to afford the pupils an opportunity for its examination: the use of the book, by pupils, during class exercise, should be entirely prohibited.

To concentrate the attention of the whole class, pupils should be called upon promiscuously to solve the problems, and not in rotation, as is too frequently the case.

No question should be read more than once, if done slowly and distinctly; the student should be required to reproduce and solve it without interruption, unless it be to make a necessary criticism or correction.

Care should be taken that the pupils (during recitation) assume an erect position and that the language they use be rigidly accurate as to construction and articulation.

Pupils, if not carefully guarded, will, in their hurried solutions, pronounce many of the most simple words incorrectly. For instance, the words: *and, of, if, for, with, what, which, where, when, costs, quarts, how many,* etc., are not unfrequently pronounced: *an, off, ef, fur, withe, wat, witch, ware, wen, coss, quats, hominy,* etc.

By careful attention to all of these particulars, a lesson in Intellectual Arithmetic is a practical lesson in elocution, grammar, rhetoric, and logic, as well as a lesson in the science of numbers."

The first owner of this book (who studied and later taught it) wrote in the front of the book: *"If at first you don't succeed, try, try again and keep on"*.

A section on Questions,Queries and Puzzles, for pupils at Home"is intriguing.

* * * * * * * * * *

("Dig That Maths" - continued from page 206)

Tigers coach made substitutions for the equation solvers and a loud explosion took place when he pulled several Trigger-nometers. At the final whisle, the Jolly Geomets led by two secants and a tangent with score Z to Y."

A few mathematical crime dramas after the approved manner of Alfred Hitchcock might do wonders to popularize math...The late, late show on TV could headline such grisly bits as: " Case of the Howling Circumference","Murder of a Mathematician","Mystery of the Missing Monomial", "Death in the Pentagon", and "Who Bisected Bessie's Obtuse Angle?" No doubt of it, math lacks suspence and the problems need zipping up with climaxes.

Math will never get mass audience appeal or make the top-ten rating without a little touch of sex...When co-eds wear bathing suits with formulas, equations and theorems printed on the curves, math classes will sport waiting lists...Juke boxes could come to the rescue with hit-parade platters: "Yes, sir, Math's my Baby", "Yes, We Have No Equation", "I Gotta Theorem, You Gotta Theorem", "I'm Trying to Figger My Sugar-Pie Out" ... Until math get a face-lifting, millions of people will never experience the exciting delight of cube roots, quadratics and the Pythagorean Theorem.

(Reprinted from the Highland Park News Herald and Journal, *Glendale California)*

* * * * * * * * * *

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

299. *Proposed by Leo Moser, University of Alberta, Canada.*

If $n = a + b + c + \dots$, prove that

$$\frac{n!}{a!b!c!\dots} \leq \frac{n^n}{a^a b^b c^c \dots}$$

300. *Proposed by J.M. Gandhi, Lingraj College, Belgaum, India.*

Three forces P, Q and R, taken in order, act along the sides of an equilateral triangle. These forces vary to become respectively L, M and N but their resultant remains the same in magnitude and direction. Prove that $(R-N)/2 = Q + M - L$.

301. *Proposed by Norman Anning, Alhambra, California.*

Prove that

$$\text{Tan } 5A = \tan A \tan (A + 36^\circ) \tan (A + 72^\circ) \tan (A + 108^\circ) \tan (A + 144^\circ).$$

302. *Proposed by C.S. Ogilvy, Hamilton College, New York.*

If a 360° arc is drawn with a pair of compasses on the surface of a cylinder and the surface is then "unrolled" and laid flat, the resulting plane curve is an oval.

1. Characterize the oval if the cylinder is a right circular cylinder.
2. Characterize the cylinder if the oval is an ellipse.

303. *Proposed by Donald A. Steinberg, University of California Radiation Laboratory.*

Let q, n be any positive, non-zero integers. Prove

1. $$\sum_{k=1}^{n} \frac{q(k-n)+k}{q+k} \binom{q+k}{k} = n$$

2. $$\sum_{k=1}^{n} \frac{q(k^2-n^2)+k(2k-1)}{q+k} \binom{q+k}{k} = n^2$$

304. *Proposed by Huseyin Demir, Kandilli Bolgesi, Turkey.*

Let ABC be a triangle, $AB \neq AC$, inscribed in a circle O, and let K be the point where the exterior angle bisector of A meets O. A variable circle with center at K meets AB, AC at E and F respectively, such that A is not an interior point of KEF. Find the limiting position m of the common point M of EF, BC as EF approaches BC.

305. *Proposed by M.S. Klamkin, Polytechnic Institute of Brooklyn.*

Find $$\lim_{z \to 1^+} (z-1) \sum_{n=0}^{\infty} \frac{2^n}{1+z^{2^n}}$$

SOLUTIONS

Errata: On page 108 of the Nov.-Dec. 1956 issue the result of problem 269 should read $(e^x - 1)(1-x)^{-2}$. On page 112, the answer to A 184 should read $I = \tan\left(\frac{n+1}{2}\right) x$.

Late Solutions

266, 269, 273. *B. Keshava R. Pai, Belgaum, India.*

276. *B. Keshava R. Pai, Belgaum, India; B. V. Torgal, Lingraj College, Belgaum, India.*

Cyclic Altitudes

278. [September 1956] *Proposed by M.N. Gopalan, Mysore, India.*

The sum of the perpendiculars drawn from the vertices of a cyclic quadrilateral $ABCD$ to the sides a, b, c, d is equal to

$$\frac{(a+b+c+d)(e+f)}{2R}$$

where e and f are the diagonals of the quadrilateral and R is the circumradius.

Solution by Leon Bankoff, Los Angeles, California. Denote the sides AB, BC, CD, DA by a, b, c, d and the diagonals AC, BD by e, f. Let

p_1, p_2 be the perpendiculars from A to BC and from B to AD. Then

$$\frac{p_1}{e} = \sin ACB = \sin ADB = \frac{p_2}{f}$$

But

$$\sin ACB = \frac{a}{2R}$$

Hence

$$\frac{p_1}{e} = \frac{p_2}{f} = \frac{a}{2R} = \frac{p_1 + p_2}{e + f}$$

In a similar manner, we find

$$\frac{p_3}{e} = \frac{p_4}{f} = \frac{b}{2R} = \frac{p_3 + p_4}{e + f}$$

where p_3 is the perpendicular from C to AB and p_4 is the perpendicular from B to CD.

Continuing this procedure, we obtain

$$\frac{\Sigma\, p_i (i = 1, 2, \ldots, 8)}{e + f} = \frac{a + b + c + d}{2R}$$

Also solved by Howard Eves, University of Maine; Chih-yi Wang, University of Minnesota and the proposer.

Integral Test Scores

279. [September 1956] *Proposed by J.M.Howell, Los Angeles City College.*

A multiple choice test with $n \geq 10$ questions and $k > 1$ choices on each question is made. If the scoring is $\left(\frac{r - w}{k - 1}\right)\left(\frac{100}{n}\right)$ where r is the number right and w the number wrong, how many tests with different values of n and k can be constructed such that all possible scores will be integers?

Solution by Francis A. C. Sevier, Rutgers College of South Jersey. It is obvious that $|r - w| \leq n$ and that $|r - w|$ assumes all integral values from 0 to n. Thus, $|100(r - w)|$ assumes all the values 0, 100, 200, 300, ..., $100n$, where $100\,n \geq 1000$ by assumption. Thus, $(k - 1)n$ must be a divisor of 100, 200, 300, ..., $100n$.

Therefore, $(k - 1)n$, under this restriction, must be 10, 20, 25, 50 or 100, i.e.,

$(k - 1)n = 10$ $(k - 1)n = 20$ $(k - 1)n = 25$ $(k - 1)n = 50$

$(k - 1)n = 100$ are admissible cases.

From these five equations, and the restriction $n \geq 10$, it is easy to see that the solutions $P(n, k)$ are, since n must be a divisor of 10, 20, 25, 50, 100:

$P_1(10,2)$, $P_2(10,3)$, $P_3(10,6)$, $P_4(10,11)$, $P_5(20,2)$, $P_6(20,6)$, $P_7(25,2)$, $P_8(25,3,)$, $P_9(25,5)$, $P_{10}(50,2)$, $P_{11}(50,3)$, $P_{12}(100,2)$.

Editor's note: Several solvers made the additional tacit assumption that all questions are answered which implies that $r + w = n$. Here the score can be written $\left(\frac{2r - n}{k - 1}\right)\left(\frac{100}{n}\right)$ If n is even then $r - w = 2r - n$ would also be even and $(k - 1)n$ must be a divisor of 200. We would have the additional solutions:

$P_{13}(10,5)$, $P_{14}(10,21)$, $P_{15}(20,3)$, $P_{16}(20,11)$, $P_{17}(40,2)$, $P_{18}(40,6)$, $P_{19}(50,5)$, $P_{20}(100,3)$ and $P_{21}(200,2)$.

Also solved by Prasert Na Nagara, College of Agriculture, Bangkok, Thailand; Robert E. Shafer, University of California Radiation Laboratory; Chih-yi Wang, University of Minnesota and the proposer.

The Mean Cevian

280. [September 1956] *Proposed by T.F. Mulcrone, St. Charles College, Louisiana.*

If the cevian AD of the acute triangle ABC is the arithmetic, (geometric), [harmonic] mean of the sides b and c of the triangle, show that $\sin \delta$, δ being the acute angle between the cevian and a, is the harmonic, (geometric), [arithmetic] mean between $\sin B$ and $\sin C$.

Solution by Sister M. Stephanie, Georgian Court College, Lakewood, New Jersey. Let $Ad = d$. In triangle ABD, $c/d = \sin(180° - \delta)/\sin B$, and in triangle ADC, $b/d = \sin\delta/\sin C$. However, since $\sin(180° - \delta) = \sin\delta$, in triangle ABD, $c/d = \sin\delta/\sin B$.

1. Arithmetic mean. Given $(b + c)/2 = d$.

Substituting values for b and c from above yields:

$$\sin\delta = \frac{2\sin B\sin C}{\sin B + \sin C}$$

This proves that $\sin\delta$ is the harmonic mean between $\sin B$ and $\sin C$.

2. Geometric mean. Given $\sqrt{bc} = d$.

Making the same substitutions as before, we find

$\sin\delta = \sqrt{\sin B\sin C}$ and the theorem is proved.

3. Harmonic mean. Given: $(2bc)/(b + c) = d$.

(Harmonic mean is reciprocal of arithmetic mean between $1/b$ and $1/c$).

The same substitutions produce:

$$\sin \delta = \frac{\sin B + \sin C}{2}.$$

This shows that $\sin \delta$ is the arithmetic mean between $\sin B$ and $\sin C$.

Also solved by Leon Bankoff, Los Angeles, California; G.W. Courter, Baton Rouge, Louisiana; Howard Eves, University of Maine; Lawrence A. Ringenberg, Eastern Illinois State College and the proposer.

Vandermonde's Determinant

281. [September 1956] *Proposed by Michael J. Pascual. Siena College, New York.*

$$\text{Let } D = \begin{vmatrix} \Sigma\, x_i^{2n} & \Sigma\, x_i^{2n-1} & \cdots & \Sigma\, x_i^{n} & \Sigma\, x_i^{n}\, y_i \\ \Sigma\, x_i^{2n-1} & \Sigma\, x_i^{2n-2} & \cdots & \Sigma\, x_i^{n-1} & \Sigma\, x_i^{n-1}\, y_i \\ \vdots & \vdots & & \vdots & \vdots \\ \Sigma\, x_i^{n} & \Sigma\, x_i^{n-1} & \cdots & (n+1) & \Sigma\, y_i \\ x^n & x^{n-1} & \cdots & 1 & y \end{vmatrix}$$

where the sums range from 0 to n. If $x_i \neq x_j$ prove that the necessary and sufficiant condition that $D = 0$ is that

$$\begin{vmatrix} x_0^n & x_0^{n-1} & \cdots & 1 & y_0 \\ x_1^n & x_1^{n-1} & \cdots & 1 & y_i \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^n & x_n^{n-1} & \cdots & 1 & y_n \\ x^n & x^{n-1} & \cdots & 1 & y \end{vmatrix} = 0$$

I. *Solution by Arthur S. Hendler, Rensselaer Polytechnic Institute, New York.* Call the second determinant defined in the statement of the problem $E(x, y, y_0, y_1, \ldots, y_n)$. Using matrix multiplication one may easily verify that

$$E'(0,1,0,0,\ldots,0)\, E(x, y, y_0, y_1, \ldots, y_n) = D$$

where E' is the transpose of E. Since $E'(0,1,0,0,\ldots,0)$ expands to a Vandermonde determinant that is not zero under the condition $x_i \neq x_j$, $D = 0$ if and only if $E(x,y,y_0,y_1,\ldots,y_n) = 0$.

II. *Solution by the proposer.* The second determinant equation represents the nth degree polynomial which passes through the $n+1$ points (x_0,y_0), (x_1,y_1), ..., (x_n,y_n). To show that the first equation also represents the same polynomial, we use the method of "least squares" to find the best relation of y to x in the form

$$y = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

corresponding to the date:

x	x_0	x_1	x_2	$\cdots$	x_n
y	y_0	y_1	y_2	$\cdots$	y_n

To find the best values for the coefficients we must minimize the function

$$f(a_0,a_1,\ldots,a_n) = \sum_{i=0}^{n} (a_n x_i^n + a_{n-1}x_i^{n-1} + \ldots + a_1 x_i + a_0 - y_i)^2$$

Using the $n+1$ equations $\dfrac{\partial f}{\partial a_i} = 0$, $i = 0, 1, \ldots, n$ along with the equation $y = a_n x^n + \ldots + a_1 x + a_0$, we find that the best such relation is given be the first determinant of the theorem. But since the smallest possible value for the sum of the squares of the deviations is zero, and in this case that value is taken on (because the assumed form can be made to fit the data exactly), then the first equation must also represent the nth degree polynomial which passes through the $n+1$ points (x_1,y_1), (x_2,y_2), ..., (x_n,y_n).

Also solved by L. Carlitz, Duke University, North Carolina: Robert E. Shafer, University of California Radiation Laboratory and Chih-yi Wang, University of Minnesota.

A Student's Error

282. [September 1956] *Proposed by Chih-yi Wang, University of Minnesota*

One student solved the problem, "Compute the area of the ellipse $x = 2\cos\theta$, $y = \sin\theta$", by using polar coordinates in the following manner:

$$\rho^2 = 4\cos^2\theta + \sin^2\theta$$

$$A = 4(\tfrac{1}{2}) \int_0^{\pi/2} (4\cos^2\theta + \sin^2\theta)\, d\theta =$$

$$4\int_0^{\pi/2}(1+\cos 2\theta)d\theta + \int_0^{\pi/2}(1-\cos 2\theta)d\theta = (5/2)\pi \text{ sq. units.}$$

As the correct answer is 2π sq.units what is wrong with his work?

I. *Solution by M. Morduchow, Polytechnic Institute of Brooklyn.* What is wrong with the work is that, contrary to what the student supposed, the parameter θ in the above equations of the ellipse is *not* the polar coordinate angle ψ (say). In fact, in polar coordinates, the equation of the given ellipse, viz. $(x/2)^2 + y^2 = 1$, would be $\rho^2 = 4/(\cos^2\psi + 4\sin^2\psi)$. Hence

$$A = 4(½)\int_0^{\pi/2}\frac{4\,d\psi}{\cos^2\psi + 4\sin^2\psi} = 2\pi$$ instead of the definite integral evaluated by the student.

II. *Solution by Robert E. Shafer, University of California Radiation Laboratory.* If we let ψ be the vectorial angle we have:

$$A = 2\int_0^{\pi/2}\rho^2\,d\psi,\quad \rho^2 = 4\cos^2\theta + \sin^2\theta \quad \text{but } \tan\psi = \frac{\sin\theta}{2\cos\theta} = \frac{\tan\theta}{2}$$

and so by differentiation we get

$$\sec^2\psi\,d\psi = \frac{\sec^2\theta}{2}\,d\theta \text{ or } d\psi = \frac{½\sec^2\theta\,d\theta}{(1+¼\tan^2\theta)} = \frac{2\,d\theta}{4\cos^2\theta + \sin^2\theta}$$

hence

$$A = 4(½)\int_0^{\pi/2} 2\left[\frac{4\cos^2\theta + \sin^2\theta}{4\cos^2\theta + \sin^2\theta}\right]d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$$

Also solved by Leon Bankoff, Los Angeles, California: Dermott A. Breault, Pittsburgh, Pennsylvania; L. Caners, St. Michaels College, Vermont; Howard Eves, University of Maine; Arthur S. Hendler, Rensselaer Polytechnic Institute, New York; Wahin Ng, San Francisco, California; Michael J. Pascual, Siena College, New York; Lawrence A. Ringenberg, Eastern Illinois State College; H. D. Ruderman, New York, New York; Sister M. Stephanie, Georgian Court College, New Jersey and the proposer.

A Multiple Summation

283. [September 1956] *Proposed by Jack Winter and Richard C. Kao, The Rand Corporation, Santa Monica, California.*

For every non negative integer n prove that

$$\sum_{a_n=0}^{n} \sum_{a_{n-1}=0}^{a_n} \sum_{a_{n-2}=0}^{a_{n-1}} \cdots \sum_{a_1=0}^{a_2} 1 = \binom{2n}{n}$$

I. *Solution by Randall M. Conkling, New Mexico College of Agriculture and Mechanic Arts*. The problem is one of iterated finite integration, involving repeated use of the formula

$\sum_{i=0}^{k} f(i) = \Delta^{-1} f(i)\Big]_0^{k+1}$ where $\Delta^{-1} f(i)$ is the function whose first forward difference over a unit interval is $f(i)$. For $\sum_{a=0}^{a_2} 1$, this formula yields $a_1\Big]_0^{a_2+1} = a_2 + 1$. Then

$$\sum_{a_2=0}^{a_3} (a_2 + 1) = \frac{a_2(a_2+1)}{2}\Big]_0^{a_3+1} = \frac{(a_3+1)(a_3+2)}{2}.$$

The next iteration yields $\dfrac{(a_4+1)(a_4+2)(a_4+3)}{3!}$, and the last iteration results in

$$\frac{(n+1)(n+2)\ \ldots\ (n+n)}{n!} \quad \text{or} \quad \binom{2n}{n}$$

II. *Solution by L. Carlitz, Duke University*. We have

$$\sum_{a_1=0}^{a_2} \binom{a_1}{k} = \binom{a_2+1}{k+1}$$

$$\sum_{a_2=0}^{a_3} \binom{a_2+1}{k+1} = \binom{a_3+2}{k+2}$$

$$\cdots$$

$$\sum_{a_n=0}^{a_{n+1}} \binom{a_n+n-1}{k+n-1} = \binom{a_{n+1}+n}{k+n} \quad \text{Thus}$$

$$\sum_{a_n=0}^{a_{n+1}} \sum_{a_{n-1}=0}^{a_n} \cdots \sum_{a_1=0}^{a_2} \binom{a_1}{k} = \binom{a_{n+1}+n}{k+n}$$

In particular for $a_{n+1} = n$, $k = 0$ this reduces to

$$\sum_{a_n=0}^{n} \sum_{a_{n-1}=0}^{a_n} \cdots \sum_{a_1=0}^{a_2} 1 = \binom{2n}{n}.$$

Also solved by Arthur S. Hendler, Rensselaer Polytechnic Institute, New York; Paul M. Pepper, Ohio State University; Robert E. Shafer, University of California Radiation Laboratory; Chih-yi Wang, University Of Minnesota and the proposers.

A Convex Envelope

284. [September 1956] *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Determine the envelope of convex polygons of n sides inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and having a maximum area.

I. *Solution by Howard Eves, University of Maine.* By an affine projection we may project the ellipse into a circle of radius a. The polygons in the ellipse then go into regular n-gons inscribed in the circle. But the envelope of all regular n-gons inscribed in a circle of radius a is a concentric circle of radius $a \cos \pi/n$. Therefore the required envelope is the ellipse

$$\frac{x^2}{(a^2\cos^2\pi/n)} + \frac{y^2}{(b^2\cos^2\pi/n)} = 1$$ similar to and concentric with the given ellipse.

II. *Solution by Chih-yi Wang, University of Minnesota.* Let the equation of the ellipse be parametrized by $x = a\cos\theta$, $y = b\sin\theta$. Let the coordinates of the vertices V_i be $(a\cos\theta_i, b\sin\theta_i)$, $i = 1, 2, 3, \ldots, n$ and in the order $\theta_{i+1} > \theta_i$ since the inscribed polygon is convex. Let the center of the ellipse be denoted by O. Then the area of the triangle OV_iV_{i+1} is $(ab)/2 \sin(\theta_{i+1} - \theta_i)$, and the area of the inscribed polygon is

$$A = \frac{ab}{2}\sum_{i=1}^{n} \sin(\theta_{i+1} - \theta_i),\ \theta_{n+1} = \theta_1 .$$

By setting $\partial A/\partial\theta_i = 0$, $i = 1, 2, \ldots, n$, we obtain the maximum value of A if $\theta_{i+1} - \theta_i = 2\pi/n$ for all i. Let α, $(0 \le \alpha < 2\pi)$ be a parameter whose geometric interpretation is an angle subtended at the center of the ellipse with the positive x-axis as one side. Then the equation of the line joining $[a\cos(\alpha + \overline{2k+1}\pi/n), b\sin(\alpha + \overline{2k+1}\pi/n)]$

and $[a \cos(\alpha + 2k\pi/n),\ b \sin(\alpha + 2k\pi/n)]$, $k = 0, 1, 2, \ldots, n-1$, is, for any fixed k.

$$F(x,y;\alpha) \equiv bx \cos\left[\alpha + \frac{(2k+1)\pi}{n}\right] + ay \sin\left[\alpha + \frac{(2k+1)\pi}{n}\right] - ab \cos\frac{\pi}{n} = 0$$

and

$$F_\alpha(x,y;\alpha) \equiv -bx \sin\left[\alpha + \frac{(2k+1)\pi}{n}\right] + ay \cos\left[\alpha + \frac{(2k+1)\pi}{n}\right].$$ By eliminating the parameter α from $F(x,y;\alpha) = 0$ and $F_\alpha(x,y;\alpha) = 0$ we get the required envelope, a similar and similarly placed ellipse

$$\frac{x^2}{a^2\cos^2(\pi/n)} + \frac{y^2}{b^2\cos^2(\pi/n)} = 1$$

As a check, if $n \to \infty$, this limiting figure is the original ellipse.

Also solved by the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 191. Find the sum of the series

$$\sum_{n=1}^{\infty} (1/n + 2/(n-1) + \ldots + n/1)x^n \quad \text{for } x < 1.$$

[*Submitted by A.K. Rajagopal*].

Q 192. Find the equation of the plane through the point (x_0, y_0, z_0) which cuts the least volume from the first octant.
[*Submitted by M. S. Klamkin*]

Q 193. Show that

$$\sum_{x=0}^{n} \frac{1}{(x!)^2[(n-x)!]^2} = \frac{(2n)!}{(n!)} .$$

[*Submitted by J. M. Howell*]

Q 194. Find the sum of the infinite series

$1/1 + 1/3 + 1/6 + \ldots$ [*Submitted by A. K. Rajagopal*].

Q 195. If $\sum_{n=1}^{\infty}(a_n)^2$ converges, prove that $\sum_{n=1}^{\infty}\frac{a_n}{n}$ converges.

[*Submitted by M. S. Klamkin*]

Q 196. If i, r are the angles of incidence and refraction for a particular wave length between any two media $(n_1 > n_2)$ whose critical angle is c, show that $i + r \leq c + 90°$ [*Submitted by A. K. Rajagopal*]

ANSWERS

A 191. The given series is the product of the power series for $-\log(1-x)$ and $(1/(1-x)^2$

A 192. Let the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Then abc is to be minimized subject to the restraint $\frac{x_0}{a}+\frac{y_0}{b}+\frac{z_0}{c}=1$ or $\left(\frac{x_0}{a}\right)\left(\frac{y_0}{b}\right)\left(\frac{z_0}{c}\right)$ is to be maximized. Since the sum of the factors is one then $\left(\frac{x_0}{a}\right)=\left(\frac{y_0}{b}\right)=\left(\frac{z_0}{c}\right)=\frac{1}{3}$. Thus the plane is $\frac{x}{3x_0}+\frac{y}{3y_0}+\frac{z}{3z_0}=1$.

A 193.

$$\sum_{x=0}^{n}\frac{1}{(x!)^2[(n-x)!]^2}=\frac{1}{(n!)^2}\sum_{x=0}^{n}\frac{(n!)^2}{(x!)^2[(n-x)!]^2}$$

$$=\frac{1}{(n!)^2}\sum_{x=0}^{n}\binom{n}{x}^2=\frac{1}{(n!)^2}\left[\frac{(2n)!}{(n!)^2}\right]=\frac{(2n)!}{(n!)^4}$$

A 194. The r^{th} term of the sequence is $T_r=1/(1+2+\ldots+r)=2/r(r+1)=2/r-2/(r+1)$. Therefore $\sum_{r=1}^{\infty}T_r=2$.

A 195. $A_n^2+\frac{1}{n^2}\geq\frac{2A_n}{n}$. But $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges. Therefore $\sum_{n=1}^{\infty}\frac{A_n}{n}$ does also.

A 196. When $i=90^\circ$ then $r=c$ so $i+r=90^\circ+c$ (extreme case). In general when $i<90$ then $r<c$ making $i+r<90+c$. Hence the theorem.